PROBLEM SET # 1

In this set $\mathfrak{g}$ is a semisimple algebra over the field of complex number and $\Delta$ is its root system.

1. Let $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. Show that
   $$(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)$$
   for any simple root $\alpha_i$.

2. Recall that a $\mathfrak{g}$-module $M$ is a weight module if $M$ is semisimple over Cartan subalgebra $\mathfrak{h}$ and all $\mathfrak{h}$-eigenspaces are finite-dimensional. Show that a submodule of a weight module is a weight module.

3. Show that the Casimir element
   $$\Omega = \sum_{i=1}^{\dim \mathfrak{g}} e_i e^i,$$
   where $\{e_i\}$ and $\{e^i\}$ are dual bases of $\mathfrak{g}$ does not depend on a choice of $\{e_i\}$ and lies in the center of the universal enveloping algebra.

4. Let $x, h, y$ be the standard basis in $\mathfrak{sl}_2$. Fix $\lambda, \mu \in \mathbb{C}$, let
   $$F_{\lambda, \mu} = t^\lambda \mathbb{C}[t, t^{-1}].$$
   Define the action of $\mathfrak{sl}_2$ on $F_{\lambda, \mu}$ by
   $$y = -\frac{\partial}{\partial t}, h = 2t \frac{\partial}{\partial t} + \mu, x = t^2 \frac{\partial}{\partial t} + \mu t.$$
   (a) Show that $F_{\lambda, \mu}$ is a weight $\mathfrak{sl}_2$-module;
   (b) Calculate the eigenvalue of the Casimir element on $F_{\lambda, \mu}$;
   (c) Find all $\lambda$ and $\mu$ for which $F_{\lambda, \mu}$ is reducible.

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