# FINAL EXAM
# MATH 261A

Start with doing Problem 0.

**0**. Fill the evaluation form for this course on website. You should get the invitation by email.

Please choose **one** of the following topics and write a detailed solution, imagining that you give a lecture on the subject. I do not mind if you use literature but you should prove all facts that were not proven in class. Please submit your work not later than **May 15**, I prefer an electronic submission and typed text if possible. Thank you. Have a great summer!

**1. Exceptional Lie groups.**

(a) Prove that the compact Lie group of type $G_2$ is isomorphic to the automorphism group of octonions.

(b) Calculate highest weights and dimensions of minimal representations (minimal dimension) of all exceptional simple Lie algebras.

(c) Give a detailed description of one exceptional Lie group of your choice, excluding $G_2$.

**2. Invariants and Molien formula.** Let $G$ be a compact Lie group and $V$ be a representation of $G$ over $\mathbb{R}$ or $\mathbb{C}$. Denote by $R$ the ring of $G$-invariant polynomials on $V$ with the natural grading.

(a) Show the following identity for the Hilbert series of $R$:

$$\sum_{n=0}^{\infty} \dim R_n t^n = \int_G \frac{1}{\det(1-tg)} dg,$$

where $dg$ is the invariant volume form on $G$ such that the volume of $G$ is 1.

(b) Let $G$ be finite. Show that $R$ is isomorphic to the polynomial ring if $G$ is generated by reflections.

(c) Compute the degrees of generators of the center of $U(\mathfrak{g})$ using the above formula for simple $\mathfrak{g}$ of rank 2.

**3. Jacobson–Morozov theorem and nilpotent cone.** Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ and $N$ denotes the cone of all nilpotent elements.

(a) For any $x \in N$ there exists a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ containing $x$.

(b) If $x \in N$, then there exists an $\mathfrak{sl}(2)$-triple $e, h, f \in \mathfrak{g}$ such that $x = e$.

(c) Show that there are finitely many $G$-orbits on $N$ with respect to the adjoint action.

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**4. Real forms of semisimple Lie algebras.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\theta$ be some real form of $\mathfrak{g}$.

(a) If $\sigma$ is the unique compact form, then $\theta$ can be chosen so that it commutes with $\sigma$.

(b) Reduce the classification of real forms to classification of involutive automorphisms. Classify real forms of all classical simple Lie algebras.

(c) If $G$ is a real Lie group with semisimple Lie algebra, then as a manifold $G = K \times \mathbb{R}^N$, where $K$ is a maximal compact Lie subgroup.

**5. Cohomology of Lie groups and Lie algebras.**

(a) Show that if $G$ is a compact Lie group, then the de Rham cohomology groups of $G$ are the same as the cohomology groups of the Lie algebra of $G$ with coefficients in the trivial module.

(b) Compute de Rham cohomology of $U(n)$.

(c) Compute $H^i(\mathfrak{gl}(n, \mathbb{C}), \mathbb{C})$.

**6. BGG resolution and Kostant's theorem.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra.

(a) Let $w \cdot \mu = w(\mu + \rho) - \rho$ denote the dot action of the Weyl group. Show that there exists a complex of Verma modules

$$0 \to M(w_0 \cdot 0) \to \cdots \to \bigoplus_{l(w)=k} M(w \cdot 0) \cdots \to M(0) \to \mathbb{C} \to 0,$$

which provides a resolution of the trivial $\mathfrak{g}$-module.

(b) For any dominant $\lambda \in P^+$ use tensoring with the simple module $L(\lambda)$ to obtain a resolution

$$0 \to M(w_0 \cdot \lambda) \to \cdots \to \bigoplus_{l(w)=k} M(w \cdot \lambda) \cdots \to M(\lambda) \to L(\lambda) \to 0$$

of $L(\lambda)$.

(c) Use this resolution to prove the Weyl character formula and to compute $H^i(\mathfrak{n}^+, L(\lambda))$.

**7. Minuscule representations.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A non-trivial simple $\mathfrak{g}$-module $L(\lambda)$ is called minuscule if all weights of $L(\lambda)$ lie on the Weyl group orbit of $\lambda$. The corresponding weight $\lambda$ is also called minuscule.

(a) If $\lambda$ is minuscule, then $\lambda$ is a fundamental weight.

(b) Prove that the number of minuscule weights equals $|P/Q| - 1$ where $P$ is the weight lattice and $Q$ is the root lattice.

(c) For every simple $\mathfrak{g}$ list all minuscule weights and compute dimensions of all minuscule representations.