PROBLEM SET # 4
MATH 251

Due October 4.

1. (Lam 5.6) For any ring R, show that the Jacobson radical of the power series ring $A = R[[t]]$ is given by

$$\operatorname{rad} A = \{a + f(t)t \mid a \in \operatorname{rad} R, f(t) \in R\}.$$

2. Give an example of a pair of rings $R \subset S$ such that $\operatorname{rad} S \cap R$ is not contained in $\operatorname{rad} R$.

3. Let V be an even-dimensional vector space over a field k and q be a non-degenerate quadratic form on V. Prove that the Clifford algebra $\operatorname{Cliff}(V, q)$ is isomorphic to a matrix algebra over some division ring.

4. (Lam 5.7) For any k-algebra R and a finite field extention $k \subset F$ show that $\operatorname{rad} R$ is nilpotent if and only if $\operatorname{rad} R^F$ is nilpotent.

5. (Lam 5.8) Let R be a $\mathbb{Z}$-graded ring, i.e.

$$R = \bigoplus_{i \in \mathbb{Z}} R_i, \;\; R_i R_j \subset R_{i+j}.$$

Let $J = \operatorname{rad} R$. Prove that J is graded i.e.

$$J = \bigoplus_{i \in \mathbb{Z}} R_i \cap J.$$

6. (Lam 6.1) Let k be a field, G be a group and H be a subgroup of finite index n. Assume that $\operatorname{char} k$ does not divide n. Let V be a $k[G]$-module. If V is semisimple over $k[H]$, then it is semisimple over $k[G]$.

7. (Lam 6.8) Let $k \subset F$ be a field extension and G be a finite group. Show that

$$\operatorname{rad}(F[G]) = \operatorname{rad}(k[G]) \otimes_k F.$$

Date: September 20, 2016.