PROBLEM SET # 3
MATH 251

Due September 20.

1. (Lam 3.9)

(a) Let R and S be rings such that $M_n(R) \simeq M_m(S)$. Does this imply $m = n$ and $S \simeq R$?

(b) Let us call a ring A a matrix ring if $A \simeq M_n(R)$ for some $n \geq 2$. True or False "A homomorphic image of a matrix ring is a matrix ring"?

2. (Lam 4.10) Show that if $f : R \to S$ is a surjective homomorphism of rings, then $f(radR) \subset radS$. Give an example to show that $f(radR)$ may be smaller than $radS$.

3. (Lam 4.13) Let R be a ring of all continuous functions on a topological space. Show that R is J-semisimple, but "in most cases" not von Neumann regular.

4. (Lam 4.17) Let R be a ring such that for every $a \in R$ the descending chain

$$Ra \supset Ra^2 \supset Ra^3 \supset \dots$$

stabilizes. Prove that R is Dedekind-finite, and every non right zero divisor is a unit.

5. Let M be a right module over a division ring D (maybe infinite-dimensional). Let $R = \mathrm{End}(M)_D$.

(a) Classify all two-sided ideals of R. (Your answer should depend on cardinality of $\dim(M_D)$. See Lam, exercise 3.16.)

(b) For any automorphism σ of R define a new R-action $*$ on on M by setting

$$a * m = \sigma(a)m.$$

Denote the corresponding module by M^σ. Prove that M^σ is isomorphic to M as an R-module. Let $\psi : M \to M^\sigma$ be this isomorphsim.

(c) Prove that there exists an automorphism τ of D such that for any $d \in D$ and $m \in M$

$$\psi(md) = \psi(m)\tau(d).$$

(d) Prove that

$$\mathrm{Aut}(R)/\,\mathrm{Inn}(R) \simeq \mathrm{Aut}(D),$$

where Aut is the group of all automorphisms and Inn is the group of inner automorphisms, i.e. all automorphisms of the form $x \mapsto uxu^{-1}$ for some unit u.

Date: September 13, 2016.