PRACTICE PROBLEMS

MATH 250A, FALL 2014

1. Let G be an abelian subgroup of the symmetric group S_n and $p_1, \ldots, p_k$ be all prime divisors of $|G|$. Prove that $n \geq p_1 + \cdots + p_k$.

2. Let p be an odd prime. Prove that for any $r > 0$ the group of units of $\mathbb{Z}/p^r\mathbb{Z}$ is cyclic.

3. Give an example of a ring R which is not isomorphic to R^{op}.

4. Let k be a field, G be a finite group and $k[G]$ denote the group ring.

(a) Any finitely generated $k[G]$-module is finite-dimensional over k.

(b) Any finite-dimensional projective module is injective.

5. Let $R = \mathbb{C}[z, z^{-1}]$ be the ring of Laurent polynomials. Prove that any finitely generated projective R-module is free.

6. Show that the primitive element theorem may not hold for a finite non-separable extension.

7. Let p be prime. Show that for any $n > 0$, there exists an irreducible polynomial in $\mathbb{F}_p[x]$ of degree n.

8. Show that any finite group is isomorphic to the Galois group of some finite extension $F \subset E$.

9. Let p be prime and ζ be a primitive p-th root of unity. Find all subfileds $F \subset \mathbb{Q}(\zeta)$ such that $[F : \mathbb{Q}] = 2$.

10. Find the Galois group of the polynomial $x^5 - 5$ over $\mathbb{Q}$.

11. Let $\bar{\mathbb{Q}} \subset \mathbb{C}$ denote the subfield of algebraic numbers and G be the (infinite) Galois group of $\bar{\mathbb{Q}}$ over $\mathbb{Q}$. We call $\alpha \in \bar{\mathbb{Q}}$ totally real if $g(\alpha) \in \mathbb{R}$ for any $g \in G$.

(a) Prove that the set H of all totally real elements is a subfiled of $\bar{\mathbb{Q}}$.

(b) Is the field extension $\mathbb{Q} \subset H$ normal?

12. Let p be a prime number and F be the splitting field for the family of polynomials $x^{p^r} - 1$ for all $r > 0$. Prove that the Galois group of F over $\mathbb{Q}$ is isomorphic to the inverse limit $\varprojlim_r U(\mathbb{Z}/p^r\mathbb{Z})$, where $U(R)$ denotes the group of units of R.