HYPERPLANE ARRANGEMENT
LECTURE NOTES

A *hyperplane* in $\mathbb{R}^d$ is the set of points $(x_1, \dots, x_d)$ satisfying the equation

$$a_0 + a_1 x_1 + \cdots + a_d x_d = 0.$$

An *arrangement* A is a finite set of hyperplanes in $\mathbb{R}^d$. A *cut* is by definition a non-empty affine subspace u which can be obtained as the intersection of some hyperplanes from A. By $L(A)$ we denote the set of all cuts. For any $u, v \in L(A)$ define $u \leq v$ if $v \subseteq u$. Then $L(A)$ is a poset with $\mathbb{R}^d$ being $\widehat{0}$. One can show easily that $L(A)$ is the lattice if and only if $L(A)$ contains $\widehat{1}$, that happens if the intersection of all hyperplanes from A is non-empty. Furthermore, $L(A)$ is graded with rank function $\rho(u) = \operatorname{codim} u$. Every interval $\left[\widehat{0}, y\right]$ in $L(A)$ is a geometric lattice, atoms are all hyperplanes $h \in A$ which contain y.

The connected components of $\mathbb{R}^d - \cup_{h \in A} h$ are called the *regions* of A. The set of regions will be denoted by $C(A)$, $c(A) = |C(A)|$.

Let $u \in L(A)$, $h \in A$, and $h \cap u \neq \varnothing$, then either $h \cap u = u$ or $h \cap u$ is a hyperplane in u. Thus, the arrangement A induces the hyperplane arrangement on u which is by definition the collection of all hyperplanes $h \cap u \subset u$. We denote this arrangement by $A \cap u$. Check that $L(A \cap u)$ is isomorphic to the subposet of all $x \in L(A)$ such that $x \geq u$.

A *face* is a region in $c \in C(A \cap u)$, dimension of a face c is by definition the dimension of u. One can check that the closure of a face is a union of faces. The set of all faces $F(A)$ is a poset with order $\alpha \leq \beta$ if α lies in the closure of β. Two arrangements A and B are *combinatorially equivalent* if $F(A)$ is isomorphic to $F(B)$. Note that $L(A) \cong L(B)$ does not imply that A and B are combinatorially equivalent.

Denote by μ_A the Moebius function in the poset $L(A)$.

Theorem 0.1. *For any hyperplane arrangement A*

$$c(A) = \sum_{x \in L(A)} \left| \mu_A\left(\widehat{0}, x\right) \right|.$$

Example 0.2. Let A be an arrangement of n hyperplanes in generic position. Then every interval $\left[\widehat{0}, x\right]$ is a Boolean lattice and

$$c(A) = \sum_{k=0}^{d} \binom{n}{k}.$$

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Proof. Denote by $A \cup h$ the arrangement obtained from A by adjoining a new hyperplane h. This new hyperplane h either does not cut a region $\alpha \in C(A)$ or divides α into two regions. The number of regions which are cut by h equals the number of regions on $A \cap h$. Therefore

$$c(A \cup h) = c(A) + c(A \cap h).$$

Now we are going to use the following

Lemma 0.3. *Let L be a geometric lattice and $\{a_0, \dots, a_n\}$ be the set of atoms. Let*

$$L' = \{x \in L \mid x = a_{i_1} \vee \dots \vee a_{i_k}, i_j \neq 0\}.$$

If $\widehat{1} \notin L'$, then

$$\mu_L\left(\widehat{0}, \widehat{1}\right) = -\mu_L\left(a_0, \widehat{1}\right).$$

If $\widehat{1} \in L'$, then

$$\mu_L\left(\widehat{0}, \widehat{1}\right) = \mu_{L'}\left(\widehat{0}, \widehat{1}\right) - \mu_L\left(a_0, \widehat{1}\right).$$

Proof. Corollary 3.9.4 in Stanley in the dual form implies

$$\mu_L\left(\widehat{0}, \widehat{1}\right) = \sum (-1)^k M_k,$$

where M_k is the number of k-tuples of atoms $\{b_1, \dots, b_k\}$ such that $\widehat{1} = b_1 \vee \dots \vee b_k$. Note that L' is geometric with atoms $\{a_1, \dots, a_n\}$ and $\left[a_0, \widehat{1}\right]$ is geometric with atoms $\{a_0 \vee a_1, \dots, a_0 \vee a_n\}$. Let M'_k is the number of k-tuples of atoms such that $\widehat{1} = a_{i_1} \vee \dots \vee a_{i_k}$ and all $i_j \neq 0$, M''_k is the number of $k-1$-tuples $a_{i_1}, \dots, a_{i_{k-1}}$ such that $\widehat{1} = a_0 \vee a_{i_1} \vee \dots \vee a_{i_{k-1}}$. Assume that $M'_k \neq 0$ at least for one k. Then

$$\mu_{L'}\left(\widehat{0}, \widehat{1}\right) = \sum (-1)^k M'_k, \ \mu_L\left(a_0, \widehat{1}\right) = \sum (-1)^{k-1} M''_k.$$

Since $M_k = M'_k + M''_k$ one obtains

$$\mu_L\left(\widehat{0}, \widehat{1}\right) = \mu_{L'}\left(\widehat{0}, \widehat{1}\right) - \mu_L\left(a_0, \widehat{1}\right).$$

If $M'_k = 0$ for all k, then

$$\mu_L\left(\widehat{0}, \widehat{1}\right) = -\mu_L\left(a_0, \widehat{1}\right).$$

∎

Now let

$$f(A) = \sum_{x \in L(A)} \left|\mu_A\left(\widehat{0}, x\right)\right|.$$

We will prove that

$$f(A \cup h) = f(A) + f(A \cap h). \tag{0.1}$$

Let $x \in L(A \cup h)$. If x does not belong to h then $\left[\widehat{0}, x\right] \subset L(A)$, hence

$$\mu_{A\cup h}\left(\widehat{0}, x\right) = \mu_A\left(\widehat{0}, x\right).$$

If x belongs to h apply Lemma 0.3 with $L = \left[\widehat{0}, x\right] \subset L(A \cup h)$, $a_0 = h$, $\{a_1, \dots, a_n\}$ being all hyperplanes in A containing x. The lemma implies

$$\mu_{A\cup h}\left(\widehat{0}, x\right) = \mu_A\left(\widehat{0}, x\right) - \mu_{A\cap h}(h, x).$$

Using

$$\mu\left(\widehat{0}, x\right)(-1)^{\rho(x)} \geq 0,$$

obtain

$$\left|\mu_{A\cup h}\left(\widehat{0}, x\right)\right| = \left|\mu_A\left(\widehat{0}, x\right)\right| + \left|\mu_{A\cap h}(h, x)\right|.$$

To get (0.1) take the sum over all $x \in L(A \cup h)$.

Since $f(A) = c(A)$ for $|A| = 1$ and $A = \varnothing$, theorem follows by induction on the number of hyperplanes and d. ☐

Denote by $C_{bd}(A)$ the set of bounded regions, put $c_{bd}(A) = |C_{bd}(A)|$.

Theorem 0.4. *Assume that the rank of an arrangement A in $\mathbb{R}^d$ equals d. Then*

$$c(A) = \left|\sum_{x\in L(A)} \mu_A\left(\widehat{0}, x\right)\right|.$$

Example 0.5. Let A be an arrangement of n hyperplanes in generic position. Then

$$c(A) = (-1)^d \sum_{k=0}^{d} (-1)^k \binom{n}{k} = \binom{n-1}{d}.$$

Proof. First, we are going to write recurrence relation for $c_{bd}(A)$.

Let h be a hyperplane in $\mathbb{R}^d$ given by the equation

$$a_0 + a_1x_1 + \cdots + a_dx_d = 0.$$

By $\widetilde{h}$ denote the hyperplane in $\mathbb{R}^{d+1}$ given by the equation

$$a_0x_0 + a_1x_1 + \cdots + a_dx_d = 0.$$

If A is a hyperplane arrangement in $\mathbb{R}^d$, set

$$\widetilde{A} = \left\{\widetilde{h} \subset \mathbb{R}^{d+1} \mid h \in A\right\}.$$

Clearly, $L\left(\widetilde{A}\right)$ is a geometric lattice. The assumption that A has rank d implies $\widehat{1} = \{0\}$. By A_P we denote the arrangement $\widetilde{A} \cup \infty$, where ∞ is the hyperplane in $\mathbb{R}^{d+1}$ defined by the equation $x_0 = 0$. Let

$$M_\infty = \left\{x \in L\left(\widetilde{A}\right) | x \subset \infty\right\}.$$

Clearly if $u \in M_\infty$ and $x \geq u \, (x \subseteq u)$, then $x \in M_\infty$. Let $\alpha \in C\,(A_P)$. There exists a unique face $\beta \in F\,(A_P \cap \infty)$ such that $\bar{\alpha} \cap \infty = \bar{\beta}$, here $\bar{\alpha}$ and $\bar{\beta}$ are the closures of α and β. Such β is called the *ideal face* of α.

Lemma 0.6. *Let β be the ideal face of some $\alpha \in C\,(A_P)$, then either $\beta \in C\,(A_P \cap \infty)$ or $\beta \in C\left(\widetilde{A} \cap u\right)$ for some $u \in M_\infty$.*

Proof. Let $\alpha \in C\,(A_P)$, then either $\alpha \in C\left(\widetilde{A}\right)$ or α is cut by the hyperplane ∞ from some $\alpha' \in C\left(\widetilde{A}\right)$. In the former case the ideal face β should belong to some cut of $\widetilde{A}$, hence $\beta \subseteq u \in M_\infty$. In the latter case β is a region of $A_P \cap \infty$. $\square$

The above proof implies the following

Corollary 0.7. *If $\beta \in C\,(A_P \cap \infty)$, then $c^\beta\,(A_P) = 2$.*

Denote by $C^\beta\,(A_P)$ the set of $\alpha \in C\,(A_P)$ whose ideal face is β and let $c^\beta\,(A_P) = |C^\beta\,(A_P)\,|$.

Lemma 0.8. $c\,(A_P) = 2c\,(A)$, $c^0\,(A_P) = 2c_{bd}\,(A)$.

Proof. For every region α of A_P there exists a unique region γ of A such that either

$$\alpha = \{(t, tx_1, \ldots, tx_d) \,|\, (x_1, \ldots, x_d) \in \gamma, t > 0\}$$

or

$$\alpha = \{(t, tx_1, \ldots, tx_d) \,|\, (x_1, \ldots, x_d) \in \gamma, t < 0\}$$

Thus, there is two-to-one correspondence between the regions of A_p and the regions of A. One can see immediately that γ is bounded iff α lies in the half space $x_0 > 0$ or $x_0 < 0$. Lemma follows. $\square$

Let $u \in M_\infty$, $\tilde{\mathrm{A}}^u = \left\{h \in \widetilde{A} \mid u \subseteq h\right\}$, define the arrangement $\widetilde{A}/u$ in the quotient space $\mathbb{R}^{d+1}/u$ as

$$\widetilde{A}/u = \left\{h/u \mid h \in \widetilde{A}^u\right\}.$$

By A/u we denote the arrangement induced by $\widetilde{A}/u$ on the image of the hyperplane $x_0 = 1$ in $\mathbb{R}^{d+1}/u$.

Lemma 0.9. *Let* $\beta \in C\left(\widetilde{A} \cap u\right)$ *for some* $u \in M_\infty$*. Then*

$$c^\beta (A_P) = c^\beta \left(\widetilde{A}\right) = c^0 \left(\widetilde{A}/u\right) = 2c_{bd} (A/u) .$$

Proof. The first equality $c^\beta (A_P) = c^\beta \left(\widetilde{A}\right)$ follows from the proof of Lemma 0.6. To prove $c^\beta \left(\widetilde{A}\right) = c^0 \left(\widetilde{A}/u\right)$ note that for any $\gamma \in C^\beta \left(\widetilde{A}\right)$ there exists exactly one $\alpha \in C\left(\widetilde{A}^u\right)$ such that $\gamma \subseteq \alpha$ and the ideal face of α is β. The natural projection $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}/u$ maps α to the region in $C\left(\widetilde{A}/u\right)$, whose ideal face is $\{0\}$. Thus, π induces the map $C^\beta \left(\widetilde{A}^u\right) \to C^0 \left(\widetilde{A}/u\right)$, (check that this map is injective and surjective). Hence

$$c^\beta \left(\widetilde{A}\right) = c^\beta \left(\widetilde{A}^u\right) = c^0 \left(\widetilde{A}/u\right) .$$

Finally, Lemma 0.8 implies $c^0 \left(\widetilde{A}/u\right) = 2c_{bd} (A/u)$. □

Corollary 0.10.

$$c (A_P) = 2c (A_P \cap \infty) + \sum_{u \in M_\infty} 2c (A_P \cap u) c_{bd} (A/u) ,$$

$$c (A) = c (A_P \cap \infty) + \sum_{u \in M_\infty} c (A_P \cap u) c_{bd} (A/u) .$$

We prove Theorem by induction on d assuming that

$$c_{bd} (A/u) = \left| \sum_{x \in L(A/u)} \mu_{A/u} \left(\widehat{0}, x\right) \right| = \left| \sum_{x \in L\left(\widetilde{A}\right), x < u, x \not\geq \infty} \mu_{\widetilde{A}} \left(\widehat{0}, x\right) \right| .$$

Note that since

$$\sum_{x \in L\left(\widetilde{A}\right)} \mu_{\widetilde{A}} \left(\widehat{0}, x\right) = 0,$$

then

$$c_{bd} (A/u) = \left| \sum_{\infty \leq x \leq u} \mu_{\widetilde{A}} \left(\widehat{0}, x\right) \right| .$$

To evaluate $c (A)$, $c (A_P \cap \infty)$ and $c (A_P \cap u)$ use Theorem 0.1

$$c (A) = \sum_{x \in L(A)} \left| \mu_A \left(\widehat{0}, x\right) \right| = \sum_{x \in L\left(\widetilde{A}\right), x \not\geq \infty} \left| \mu_{\widetilde{A}} \left(\widehat{0}, x\right) \right| = \sum_{x \in L(A_P), x \not\geq \infty} \left| \mu_{A_P} \left(\widehat{0}, x\right) \right| ,$$

$$c (A_P \cap \infty) = \sum_{x \in L(A_P), x \geq \infty} \left| \mu_{A_P} (\infty, x) \right| ,$$

$$c\left(A_P \cap u\right) = \sum_{x \in L(A_P), x \geq u} \left|\mu_{A_P}\left(u, x\right)\right|$$

Thus, to prove Theorem it suffices to verify the identity

$$\sum_{x \ngeq \infty} \left|\mu_{A_P}\left(\widehat{0}, x\right)\right| = \sum_{x \geq \infty} \left|\mu_{A_P}\left(\infty, x\right)\right| + \sum_{u \in M_\infty} \left|\sum_{\infty \leq x \leq u} \mu_{\widetilde{A}}\left(\widehat{0}, x\right)\right| \sum_{y \geq u} \left|\mu_{A_P}\left(u, y\right)\right|. \tag{0.2}$$

Rewrite

$$\sum_{u \in M_\infty} \left|\sum_{\infty \leq x \leq u} \mu_{\widetilde{A}}\left(\widehat{0}, x\right)\right| \sum_{y \geq u} \left|\mu_{A_P}\left(u, y\right)\right| = \sum_{\infty \leq x \leq u \leq y} (-1)^{\rho(u)} \mu_{\widetilde{A}}\left(\widehat{0}, x\right) (-1)^{\rho(y)-\rho(u)} \mu_{A_P}\left(u, y\right).$$

The relations

$$\sum_{x \leq u \leq y} \mu_{A_P}\left(u, y\right) = 0$$

if $x \neq y$ and

$$\sum_{x \leq u \leq y} \mu_{A_P}\left(u, y\right) = 1$$

if $x = y$ imply

$$\sum_{u \in M_\infty} \left|\sum_{\infty \leq x \leq u} \mu_{\widetilde{A}}\left(\widehat{0}, x\right)\right| \sum_{y \geq u} \left|\mu_{A_P}\left(u, y\right)\right| = \sum_{x \in M_\infty} \left|\mu_{\widetilde{A}}\left(\widehat{0}, x\right)\right|.$$

Now (0.2) becomes

$$\sum_{x \ngeq \infty} \left|\mu_{A_P}\left(\widehat{0}, x\right)\right| = \sum_{x \geq \infty} \left|\mu_{A_P}\left(\infty, x\right)\right| + \sum_{x \in M_\infty} \left|\mu_{\widetilde{A}}\left(\widehat{0}, x\right)\right|. \tag{0.3}$$

Let

$$N_x = \left\{x \in L\left(\widetilde{A}\right) | x \cap \infty \notin L\left(\widetilde{A}\right)\right\}, \; O_x = \left\{x \in L\left(\widetilde{A}\right) | x \cap \infty \in L\left(\widetilde{A}\right)\right\}.$$

Check that if $x \in N_\infty$, then the intervals $\left[\widehat{0}, x\right]$ and $[\infty, x \vee \infty]$ are isomorphic, isomorphism is defined by $y \to y \vee \infty$. Therefore (0.3) can be reduced to

$$\sum_{x \in O_\infty} \left|\mu_{A_P}\left(\widehat{0}, x\right)\right| = \sum_{x \in M_\infty} \left|\mu_{A_P}\left(\infty, x\right)\right| + \sum_{x \in M_\infty} \left|\mu_{\widetilde{A}}\left(\widehat{0}, x\right)\right|, \tag{0.4}$$

using

$$\sum_{x \ngeq \infty} \left|\mu_{A_P}\left(\widehat{0}, x\right)\right| = \sum_{x \in O_\infty} \left|\mu_{A_P}\left(\widehat{0}, x\right)\right| + \sum_{x \in N_\infty} \left|\mu_{A_P}\left(\widehat{0}, x\right)\right|,$$

$$\sum_{x \geq \infty} \left|\mu_{A_P}\left(\infty, x\right)\right| = \sum_{x \in M_\infty} \left|\mu_{A_P}\left(\infty, x\right)\right| + \sum_{x \in N_\infty} \left|\mu_{A_P}\left(\infty, x \vee \infty\right)\right|$$

Lemma 0.11.

$$\sum_{x\in O_\infty}\left|\mu_{A_P}\left(\widehat{0},x\right)\right|=\sum_{x\in M_\infty}\left|\mu_{A_P}\left(\widehat{0},x\right)\right|.$$

Proof. Use formula (27) section 3.10 in Stanley for the lattice $\left[\widehat{0},x\right]$, $x\in M_\infty$. The coatoms are ∞ and all $y\in O_\infty$ such that $y\vee\infty=y\cap\infty=x$. Thus,

$$\left|\mu_{A_P}\left(\widehat{0},x\right)\right|=\sum_{y\in O_\infty,\ y\vee\infty=x}\left|\mu_{A_P}\left(\widehat{0},y\right)\right|$$

for any $x\in M_\infty$. Now take the sum over all $x\in M_\infty$ and obtain the desired formula. □

Thus, (0.4) is equivalent to

$$\sum_{x\in M_\infty}\left|\mu_{A_P}\left(\widehat{0},x\right)\right|=\sum_{x\in M_\infty}\left|\mu_{A_P}\left(\infty,x\right)\right|+\sum_{x\in M_\infty}\left|\mu_{\widetilde{A}}\left(\widehat{0},x\right)\right|.$$

But Lemma 0.3 implies

$$\left|\mu_{A_P}\left(\widehat{0},x\right)\right|=\left|\mu_{A_P}\left(\infty,x\right)\right|+\left|\mu_{\widetilde{A}}\left(\widehat{0},x\right)\right|$$

for every $x\in M_\infty$. The proof of Theorem is complete. □

Let $S\subseteq A$ and $u=\cap_{h\in S}h\neq\varnothing$. The *nullity* of S is the number $n\left(S\right)=\left|S\right|-\rho\left(u\right)$. Let d_k be the number of $S\subseteq A$ such that $n\left(S\right)=k$.

Theorem 0.12.

$$c\left(A\right)=\sum\left(-1\right)^k d_k.$$

Proof. For any $x\in L\left(A\right)$ let

$$S_x=\left\{S\subseteq A\mid x=\cap_{h\in S}h\right\}.$$

Then By Corollary 3.9.4 in Stanley (in dual form)

$$\mu\left(\widehat{0},x\right)=\sum_{s\in S_x}\left(-1\right)^{|S|},$$

$$\sum_{x\in L(A)}\left|\mu\left(\widehat{0},x\right)\right|=\sum_{x\in L(A)}\left(-1\right)^{\rho(x)}\mu\left(\widehat{0},x\right)=\sum_{x\in L(A),S\in S_x}\left(-1\right)^{\rho(x)+|S|}=\sum_{S\subseteq A}\left(-1\right)^{n(S)}.$$

□

Let $f_k\left(A\right)$ denote the number of faces of dimension k and $f_k^{bd}\left(A\right)$ denote the number of bounded faces of dimension k.

Theorem 0.13. *For any arrangement A*

$$\sum (-1)^k f_k(A) = (-1)^d,$$

and

$$\sum (-1)^k f_k^{bd}(A) = 1$$

if rank of A is d.

Proof. Theorem 0.1 implies

$$f_k(A) = \sum_{u \in L(A),\ \rho(u)=d-k, x \geq u} |\mu_A(u,x)|.$$

Hence

$$\sum (-1)^k f_k(A) = \sum_{x \geq u} (-1)^{d-\rho(u)} (-1)^{\rho(x)-\rho(u)} \mu_A(u,x) = (-1)^d.$$

Theorem 0.4 implies

$$f_k^{bd}(A) = \sum_{u \in L(A), \rho(u)=d-k} \left| \sum_{x \geq u} \mu_A(u,x) \right| = \sum_{x \geq u} (-1)^{d-\rho(u)} \mu_A(u,x).$$

Therefore

$$\sum (-1)^k f_k^{bd}(A) = \sum_{x \geq u} \mu_A(u,x) = 1.$$

□

Reference: Zaslavsky, Mem. Amer. Math. Society, no. 154.