PROBLEM SET # 11
MATH 249

Due November 21.

1. Let J_N denote the ideal in $\mathbb{C}[x_1,\dots,x_N]$ generated by elementary symmetric polynomials $e_1,\dots,e_N$. Define $R_N=\mathbb{C}[x_1,\dots,x_N]/J_N$. Show that R_N is a finite-dimensional algebra over $\mathbb{C}$. Prove the inequality

$$\dim R_N \le N!.$$

Hint: note that x_1 is a root of a monic polynomial of degree N with coefficients $\pm e_i$ and use induction on N.

2. Let $s_i=(i,i+1)$ with $i\le N-1$. Define the linear operator $D_i:\mathbb{C}[x_1,\dots,x_N]\to\mathbb{C}[x_1,\dots,x_N]$ by the formula

$$D_i(f)=\frac{f(s_ix)-f(x)}{x_{i+1}-x_i}.$$

Prove that D_i is well-defined, i.e. $D_i(f)$ is a polynomial, and that $D_i(J_N)\subset J_N$. Thus, D_i is a well-defined operator on R_N.

3. Show that $D_iD_j=D_jD_i$ if $i\neq j\pm1$ and $D_iD_{i+1}D_i=D_{i+1}D_iD_{i+1}$. Using this prove that if $s_{i_1}\dots s_{i_k}=s_{j_1}\dots s_{j_k}=w$ with $k=l(w)$, then

$$D_{i_1}\dots D_{i_k}=D_{j_1}\dots D_{j_k}.$$

Thus, for each element $w\in W$ one can define the operator D_w which does not depend on a reduced decomposition of w into the product of adjacent transpositions. You may use without proof that the relations $s_is_j=s_js_i$, for $i\neq j\pm1$, $s_is_{i+1}s_i=s_{i+1}s_is_{i+1}$ and $s_i^2=1$ generate S_N.

4. Let $p_w=D_w\left(x_N^{N-1}x_{N-1}^{N-2}\dots x_2\right)$. Prove that $\{p_w\}_{w\in S_N}$ is a basis in R_N. Hence $\dim R_N=N!$. The polynomials p_w are called Schubert polynomials.

Date: November 14, 2006.