Due November 14.

1. Let

\[ P = \left\{ (x_1, \ldots, x_N) \in \mathbb{R}^N \mid 0 \leq x_i \leq 1, \sum_{i=1}^{N} x_i = 1 \right\} . \]

Show that \( P \) is a convex polytope and find \( i(P, n) \) and \( i^+(P, n) \).

2. Let \( P \) be a convex integral polytope in \( \mathbb{R}^N \) and \( f : \mathbb{R}^N \to \mathbb{R}^N \) be defined by the formula

\[ f(x) = Ax + \beta, \]

where \( \beta \in \mathbb{Z}^N \), \( A \) be a matrix with integral coefficients such that \( \det A = \pm 1 \). Prove that \( i(P, n) = i(f(P), n) \).

3. Let \( \text{Par}(n) \) denote the set of partitions of \( n \). Define a partial order on \( \text{Par}(n) \) by \( \lambda \leq \mu \) if \( \lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k \) for all \( k \geq 1 \). Prove that \( \text{Par}(n) \) is a lattice.

4. Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \text{Par}(n) \) and \( P_\lambda \) be the convex hull of \( (\lambda_{s(1)}, \ldots, \lambda_{s(n)}) \) for all \( s \in S_n \). Prove that \( \lambda \leq \mu \) iff \( P_\lambda \subseteq P_\mu \).