Due December 7.
Choose five problems from the list below.

1. Give a combinatorial proof for the following identities

\[
S(n + 1, m + 1) = \sum_{k=0}^{n} \binom{n}{k} S(k, m),
\]

\[
s(n + 1, m + 1) = \sum_{k=m}^{n} s(n, k) \binom{k}{m},
\]

here \( S(n, l) \) is the Stirling number of the second kind, \( s(n, l) \) is the signless Stirling number of the first kind.

2. Give a combinatorial proof of the identity

\[
\prod_{i \geq 0} (1 + tx^{2i+1}) = \sum_{k \geq 0} x^k \frac{e^{x(k+1)}}{(1-x^2)(1-x^4)\ldots(1-x^{2k})}.
\]

3. In how many ways one can sit \( n \) couples around the table so that nobody sits near his own spouse?

4. Let \( P \) be the convex hull of \((s(1), \ldots, s(n)) \in \mathbb{R}^n\) for all \( s \in S_n \). Find \( f \) and \( h \)-polynomials for \( P \). Recall that \( f(x) = \sum f_k x^k \), where \( f_k \) is the number of \( k \)-dimensional faces, \( h(x) = f(x-1) \).

5. Let \( P \) be an integral convex polytope in \( \mathbb{R}^d \) of dimension \( d \). Prove that the volume of \( P \) can be calculated by the formula

\[
\frac{1}{d!} \sum_{k=0}^{d} \binom{d}{k} (-1)^{d-k} i(P, k),
\]

where \( i(P, k) \) is the Ehrhart polynomial.

6. Let \( n \) be a positive integer. Let \( \mu(n) = 0 \) if \( p^2|n \) for some prime \( p \) and \( \mu(n) = (-1)^{k(n)} \) where \( k(n) \) is the number of prime factors of \( n \) if \( p^2 \) does not divide \( n \) for all prime \( p \).
   (a) For a given sequence \( f(n) \) let

\[
g(n) = \sum_{t|n} f(t).
\]
Show that

\[ f(n) = \sum_{t \mid n} \mu\left(\frac{n}{t}\right) g(t). \]

(b) Prove the identity

\[ \prod_{n=1}^{\infty} \left(1 - x^n\right)^{-\mu(n)/n} = e^x. \]

7. Let \( K_5 \) be the complete graph with 5 vertices. Prove that it is impossible to embed \( K_5 \) into a sphere. Find a surface of a minimal genus in which \( K_5 \) can be embedded.

8. Prove the following identity for symmetric functions

\[ \sum_{i=0}^{k} (-1)^i h_{k-i} e_{l+i} = s_{\lambda}, \]

where \( \lambda = (k + 1, 1^{l-1}) \).