# FINAL EXAM
# MATH 249

Due December 7.

Choose five problems from the list below.

**1**. Give a combinatorial proof for the following identities

$$S(n+1,m+1)=\sum_{k=0}^{n}\binom{n}{k}S(k,m),$$

$$s(n+1,m+1)=\sum_{k=m}^{n}s(n,k)\binom{k}{m},$$

here $S(n,l)$ is the Stirling number of the second kind, $s(n,l)$ is the signless Stirling number of the first kind.

**2**. Give a combinatorial proof of the identity

$$\prod_{i\geq 0}\left(1+tx^{2i+1}\right)=\sum_{k\geq 0}\frac{x^{k^2}t^k}{(1-x^2)(1-x^4)\dots(1-x^{2k})}.$$

**3**. In how many ways one can sit $n$ couples around the table so that nobody sits near his own spouse?

**4**. Let $\mathcal{P}$ be the convex hull of $(s(1),\dots,s(n))\in\mathbb{R}^n$ for all $s\in S_n$. Find $f$ and $h$-polynomials for $\mathcal{P}$. Recall that $f(x)=\sum f_k x^k$, where $f_k$ is the number of $k$-dimensional faces, $h(x)=f(x-1)$.

**5**. Let $\mathcal{P}$ be an integral convex polytope in $\mathbb{R}^d$ of dimension $d$. Prove that the volume of $\mathcal{P}$ can be calculated by the formula

$$\frac{1}{d!}\sum_{k=0}^{d}\binom{d}{k}(-1)^{d-k}\,i(\mathcal{P},k),$$

where $i(\mathcal{P},k)$ is the Ehrhart polynomial.

**6**. Let $n$ be a positive integer. Let $\mu(n)=0$ if $p^2|n$ for some prime $p$ and $\mu(n)=(-1)^{k(n)}$ where $k(n)$ is the number of prime factors of $n$ if $p^2$ does not divide $n$ for all prime $p$.

(a) For a given sequence $f(n)$ let

$$g(n)=\sum_{t|n}f(t).$$

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Show that

$$f(n)=\sum_{t|n}\mu\left(\frac{n}{t}\right)g(t).$$

($b$) Prove the identity

$$\prod_{n=1}^{\infty}(1-x^n)^{\frac{-\mu(n)}{n}}=e^x.$$

**7**. Let $K_5$ be the complete graph with 5 vertices. Prove that it is impossible to embed $K_5$ into a sphere. Find a surface of a minimal genus in which $K_5$ can be embedded.

**8**. Prove the following identity for symmetric functions

$$\sum_{i=0}^{k}(-1)^i h_{k-i}e_{l+i}=s_\lambda,$$

where $\lambda=\left(k+1,1^{l-1}\right)$.