Math 249 problem set 5

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1 Yes, every such matrix is the Laplace matrix of some graph. Given L we can construct a loopfree graph Γ to have L as its Laplace matrix. Let $V(\Gamma) = [n]$, where n is the dimension of L. For each pair i, j of distinct vertices, let Γ contain exactly $-l_{ij} = -l_{ji}$ edges between i and j; this is possible by symmetry of L and since this entry is a nonnegative integer. Now, since the sum of each row is zero, the diagonal entry l_{ii} of row i is the sum of the negatives of the other entries, so it's exactly the degree of vertex i, as required.

2 Since L is a real symmetric matrix, all its eigenvalues are real, so we need only eliminate the negative and too large positive cases. Label the vertices of Γ by $[n]$, where $n = |V|$.

Suppose $\mathbf{v} = (v_i)$ were a nonzero eigenvector corresponding to a negative eigenvalue λ; by negating we may assume $\mathbf{v}$ has a positive component. Without loss of generality let v_1 be a maximal positive component of $\mathbf{v}$, so that the first component of $L\mathbf{v}$ is λv_1, which is negative. However, this first component is $\sum_j l_{1j} v_j = \sum_{j \neq 1} l_{1j}(v_j - v_1)$ by the zero-sum property, and each $v_j - v_1$ is nonpositive by assumption, as is each l_{1j}, so this sum is nonnegative, contradiction. So L has no negative eigenvalues.

Now suppose $\mathbf{v} = (v_i)$ were a nonzero eigenvector corresponding to an eigenvalue λ such that $\lambda > 2 \deg v$ for each vertex v of G. This time assume without loss of generality that v_1 is a positive component of maximal absolute value among all the components. The first component of $L\mathbf{v}$ is $\lambda v_1 \geq 2\deg(1) v_1$. But again we may express this as $\sum_j l_{1j} v_j = \sum_{j \neq 1} l_{1j}(v_j - v_1) = \sum_{e \in E_1}(v_j - v_1)$ where E_1 is the set of all edges incident to the vertex 1. By choice of v_1 each $v_j - v_1 \leq 2v_1$, so $\sum_{e \in E_1} v_j - v_1 \leq 2v_1 \deg(1)$, contradiction.

3 For any $i \neq j$, there is an edge of Γ joining each of $\pm e_i$ to $\pm e_j$; the only pairs of vertices in Γ joined by no edge are $e_i, -e_i$ for each i. The total degree of each vertex is therefore $2(n-1)$.

The graph Γ has $2n$ vertices. so its complexity will be given by $\prod_{i=1}^{2n-1} \mu_i/(2n)$ where the μ_i are a collection of eigenvalues of the Laplace matrix L with multiplicity, omitting a single eigenvalue 0. To form L we'll order the vertices so that each pair $e_i, -e_i$ are consecutive. Then, by our observations above, the diagonal entries of L are $2(n-1)$, the off-diagonal entries $l_{2k-1,2k} = l_{2k,2k-1}$ in each 2 by 2 block are 0, and the other entries are -1.

Now, to find the eigenvalues of L, observe that $L-2(n-1)I_{2n}$ is the Kronecker product $1_n - I_n \otimes -1_2$, where 1_l is the l by l matrix all of whose entries are 1. The eigenvalues of 1_n are $n, 0^{n-1}$, so those of $1_n - I_n$ are $n-1, (-1)^{n-1}$, and the eigenvalues of -1_2 are -2 and 0, so the eigenvalues of their Kronecker product are the collection of pairwise products of these, or $-2(n-1), 2^{n-1}, 0^n$. Thus the eigenvalues of L are $0, (2n)^{n-1}, (2(n-1))^n$, and we conclude that the complexity of Γ is $2n^{n-1}(2(n-1))^n/(2n) = 2^{2n-2}n^{n-2}(n-1)^n$.

4 If we've glued a $2n$-gon to obtain a torus, which has genus 1, then we'll obtain something with n edges and one face on this surface, so by Euler's formula $|V| - n + 1 = 0$, i.e. $V = n - 1$.

Now, we invoke the form of the Harer-Zagier formula that states

$$T_n(x) = \frac{(2n)!}{2^n n!} \sum_l 2^{l-1} \binom{x}{l} \binom{n}{l-1}.$$

We are looking for $[x^{n-1}]T_n(x)$. The binomial coefficient $\binom{x}{l}$ can only generate an x^{n-1} term when $l \geq n-1$, and $\binom{n}{l-1}$ will only be nonzero when $n \geq l-1$. So we have three terms $l = n-1, n, n+1$ to worry about. Now $\binom{x}{l} = (\prod_{i=0}^{l-1} x - i)/l!$; so we have

$$l![x^l]\binom{x}{l} = 1,$$

$$-l![x^{l-1}]\binom{x}{l} = \sum_{i=0}^{l-1} i = \binom{l}{2},$$

$$l![x^{l-2}]\binom{x}{l} = \sum_{i=0}^{l-1}\sum_{j=0}^{i-1} ij = \sum_{i=0}^{l-1} i\binom{i}{2} = \sum_{i=0}^{l-1}\left(3\binom{i}{3} + 2\binom{i}{2}\right) = 3\binom{l}{4} + 2\binom{l}{3}.$$

We therefore have

$$\begin{aligned}[x^{n-1}]T_n(x) &= \frac{(2n)!}{2^n n!} 2^{n-2} \left(\frac{\binom{n}{n-2}}{(n-1)!} - \frac{2\binom{n}{2}\binom{n}{n-1}}{n!} + \frac{4\left(3\binom{n+1}{4} + 2\binom{n+1}{3}\right)\binom{n}{n}}{(n+1)!} \right) \\ &= \frac{(2n)!}{n!(n+1)!} \frac{n^3-n}{12} = c_n \frac{n^3-n}{12}.\end{aligned}$$