# PRACTICE MIDTERM SOLUTIONS.
# MATH $1B$

## 1. Test 1

**1.**

$$ds = \sqrt{1 + ((5x^{\frac{3}{2}})')^2}dx = \sqrt{1 + \frac{225x}{4}}dx.$$

The length of the curve is given by the integral

$$\int_1^2 \sqrt{1 + \frac{225x}{4}}dx = \frac{4}{225}\frac{2}{3}\left(1 + \frac{225x}{4}\right)^{\frac{3}{2}}|_1^2 = \frac{454^{\frac{3}{2}} - 229^{\frac{3}{2}}}{675}.$$

**2.**
(a)

$$\lim_{n\to\infty} \frac{\ln(2n)}{\ln n} = \lim_{n\to\infty} \frac{\ln 2 + \ln n}{\ln n} = \lim_{n\to\infty} \frac{\ln 2}{\ln n} + 1 = 1.$$

(b)

$$0 \le \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot \dots \cdot n}{n^n} \le \frac{1}{n}.$$

Therefore

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

by the Squeeze Theorem.
**3.**
(a) Use the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n+n^2}}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sqrt{n+n^2}}{n} = \lim_{n\to\infty} \sqrt{\frac{1}{n} + 1} = 1.$$

The series is convergent.
(b) The series is divergent by divergence test.

$$\lim_{n\to\infty} \ln \frac{n^2+1}{2n^2} = \ln\left(\frac{1}{2} + \frac{1}{2n^2}\right) = \ln\frac{1}{2} \ne 0.$$

**4.** Use the ratio test

$$\lim_{n\to\infty} \frac{\frac{|x-1|^{3n+3}}{8^{n+1}(n+1)^2}}{\frac{|x-1|^{3n}}{8^n n^2}} = \frac{|x-1|^3}{8}.$$

The series converges for $\frac{|x-1|^3}{8} < 1$, that implies $|x-1| < 2$. Therefore the radius of convergence is 2. The endpoints of the interval are $-1$ and 3. If $x = 3$ the series equals

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

If $x = -1$ the series equals

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

Both series converge. Therefore the interval of convergence of the power series is $[-1, 3]$.

**5**. Using partial fractions we obtain

$$\frac{1}{x^2+5x+4} = \frac{1}{3}\left(\frac{1}{x+1} - \frac{1}{x+4}\right).$$

From geometric series formulas we get

$$\frac{1}{x+1} = \sum_{n=0}^{\infty} (-1)^n x^n$$

and

$$\frac{1}{x+4} = \frac{1}{4}\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{4^n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{4^{n+1}}.$$

Therefore

$$\frac{1}{x^2+5x+4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3}\left(1 - \frac{1}{4^{n+1}}\right) x^n.$$

## 2. Test 2

**1**.
(a) The average value $\mu$ can be found by the formula

$$\mu = \int_0^\infty tce^{-ct}dt = \frac{1}{c}.$$

Therefore $c = \frac{1}{\mu} = \frac{1}{500}$.
(b)

$$P(t \leq 200) = \int_0^{200} \frac{e^{-t/500}}{500}dt = 1 - e^{-2/5} \sim 2/5.$$

(c)

$$\frac{1}{2} = \int_0^t \frac{e^{-t/500}}{500}dt = 1 - e^{-t/500},$$

$$e^{-t/500} = \frac{1}{2}, \ \frac{t}{500} = \ln 2, \ t = 500 \ln 2.$$

**2**. Find the limits
(a)

$$-\frac{1}{\sqrt{n}} \leq \frac{\sin n}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}.$$

Therefore the limit is zero by the Squeeze Theorem.
(b)

$$\lim_{n\to\infty} \ln\left((1 - \frac{1}{n})^n\right) = \lim_{n\to\infty} n \ln\left(1 - \frac{1}{n}\right) = \lim_{x\to 0} \frac{\ln(1-x)}{x} = \lim_{x\to 0} \frac{-1}{1-x} = -1.$$

Therefore

$$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = e^{-1}.$$

**3**. (a) We use the alternating series test

$$\sum_{n=1}^\infty \frac{\cos(\pi n)}{\sqrt{n}} = \sum_{n=1}^\infty \frac{(-1)^n}{\sqrt{n}}.$$

We know that $\frac{1}{\sqrt{n}}$ is positive and decreasing. Hence the series converges.
(b) Use the ratio test

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2n^3}{(n+1)^3} = 2 > 1.$$

The series is divergent.
**4**. Consider the power series

$$\sum_{n=1}^\infty nx^n.$$

Note that

$$\sum_{n=1}^{\infty} nx^n = x(\sum_{n=0}^{\infty} x^n)' = x\left(\frac{1}{1-x}\right)' = \frac{x}{(1-x)^2}.$$

Substitute $x = \frac{1}{3}$ and get the answer $\frac{3}{4}$.

**5**. Use the binomial power series

$$(1+y)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} y^n.$$

First use the substitution $y = 4x^2$ to get

$$\frac{1}{\sqrt{1+4x^2}} \sum_{n=0}^{\infty} \binom{-1/2}{n} 4^n x^{2n}.$$

Then multiply by $x$

$$f(x) = \frac{x}{\sqrt{1+4x^2}} = \sum_{n=0}^{\infty} \binom{-1/2}{n} 4^n x^{2n+1}.$$