SOLUTIONS FOR PRACTICE MIDTERMS

Test 1.
1.
(a) Use the substitution $u = \sin x$.

$$\int \sin 2x \sin x\, dx = \int 2\sin^2 x \cos x\, dx = \int 2u^2 du = \frac{2}{3}u^3 + C = \frac{2}{3}\sin^3 x + C.$$

$$(b) \int \frac{dt}{t^2 + t^3} = \int \left(\frac{1}{t^2} - \frac{1}{t} + \frac{1}{t+1}\right) dt = -\frac{1}{t} + \ln|\frac{t+1}{t}| + C$$

(**c**) Make substitution $u = x^{1/4}$, $x = u^4$, $dx = 4u^3 du$

$$\int \frac{x^{1/2}dx}{1 + x^{1/4}} = 4\int \frac{u^5 du}{1+u} = 4\int \left(u^4 - u^3 + u^2 - u + 1 - \frac{1}{1+u}\right) du =$$

$$\frac{4}{5}u^5 - u^4 + \frac{4}{3}u^3 - 2u^2 + 4u - 4\ln|1+u| + C =$$

$$\frac{4}{5}x^{5/4} - x + \frac{4}{3}x^{3/4} - 2x^{1/2} + 4x^{1/4} - 4\ln|1 + x^{1/4}| + C$$

(**d**) Use the substitution $x = u^2$, $dx = 2u\, du$

$$\int \cos\sqrt{x}\, dx = 2\int u\cos u\, du = 2u\sin u - 2\int \sin u\, du = 2u\sin u + 2\cos u + C =$$

$$2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$$

2.

$$(a) \int_0^1 \frac{dx}{x^3} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x^3} = \lim_{t\to 0^+}\left(-\frac{1}{2} + \frac{1}{2t^2}\right) = \infty.$$

The integral is divergent.
(b)

$$\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C;$$

$$\int_0^\infty xe^{-x}dx = \lim_{t\to\infty}\left(-te^{-t} - e^{-t} + 1\right) = 1$$

3. The second derivarive

$$|\left(e^{1/x}\right)''| = |\left(\frac{1}{x^4} + \frac{2}{x^3}\right) e^{1/x}| \le 3e \le 8.16$$

Using the estimation formula for an error one has

$$\frac{8.16}{12n^2} < 0.01,\ n^2 > \frac{816}{12},\ n = 9.$$

4. Rewrite the integral using the substitution $u = x + 1$

$$\int_{-\infty}^{\infty} e^{-x^2-2x} dx = e \int_{-\infty}^{\infty} e^{-x^2-2x-1} dx = e \int_{-\infty}^{\infty} e^{-u^2} du$$

The integral is convergent, see example 9, section 7.8.

Test 2.

1.

(a) Use $u = \sin x$

$$\int \sin^3 x \cos x \, dx = \int u^3 du = \frac{1}{4} u^4 + C = \frac{1}{4} \sin^4 x + C$$

(b) substitution $u = x - 1$

$$\int \frac{x^3}{(x-1)^{13}} dx = \int \frac{(u+1)^3}{u^{13}} du = \int \frac{u^3 + 3u^2 + 3u + 1}{u^{13}} du =$$

$$\int \left(u^{-10} + 3u^{-11} + 3u^{-12} + u^{-13}\right) du = -\frac{1}{9} u^{-9} - \frac{3}{10} u^{-10} - \frac{3}{11} u^{-11} - \frac{1}{12} u^{-12} + C =$$

$$-\frac{1}{9}(x-1)^{-9} - \frac{3}{10}(x-1)^{-10} - \frac{3}{11}(x-1)^{-11} - \frac{1}{12}(x-1)^{-12} + C$$

(**c**) Use $u = e^x$, $dx = \frac{du}{u}$

$$\int \frac{dx}{e^{2x} + 3e^x + 2} = \int \frac{du}{u\left(u^2 + 3u + 2\right)} = \int \left(\frac{1}{2u} - \frac{1}{u+1} + \frac{1}{2(u+2)}\right) du =$$

$$\frac{1}{2} \ln |u| - \ln |u+1| + \frac{1}{2} \ln |u+2| + C = \frac{1}{2} x - \ln(1 + e^x) + \frac{1}{2} \ln(2 + e^x) + C$$

(**d**) Use the trigonometric substitution $x = 2\sin\theta$, $dx = 2\cos\theta \, d\theta$, $4 - x^2 = 4\cos^2\theta$

$$\int \left(4 - x^2\right)^{1/2} dx = 4 \int \cos^2 \theta d\theta = 4 \int \frac{1 + \cos 2\theta}{2} d\theta = 2\theta + \sin 2\theta + C =$$

$2 \text{ arc } \sin \frac{x}{2} + \frac{x}{2}\left(4 - x^2\right)^{1/2} + C$

2. Using integration by parts

$$I_n = \int_0^{\pi/2} \sin^n x dx = -\sin^{n-1} x \cos x|_0^{\pi/2} + (n-1) \int_0^{\pi/2} \sin^{n-2} x \cos^2 x dx.$$

Using $\cos^2 x = 1 - \sin^2 x$ get

$$I_n = (n-1) \int_0^{\pi/2} \left(\sin^{n-2} x - \sin^n x\right) dx = (n-1) I_{n-2} - (n-1) I_n.$$

Therefore

$$I_n = \frac{n-1}{n} I_{n-2}.$$

3. Use the following estimation for the second derivative

$$|\left(\cos x^2\right)''| = |-4x^2 \cos x^2 - 2\sin x^2| < 402,$$

and the formula for E_M gets

$$\frac{10^3 \times 402}{24n^2} \leq 10^{-5},\ n^2 \geq 2 \times 10^9,\ n > 4.5 \times 10^4.$$

4. By definition

$$\int_0^{\pi/2} \sec x\, dx = \lim_{t \to \frac{\pi}{2}^-} \int_0^t \sec x\, dx =$$

$$\lim_{t \to \frac{\pi}{2}^-} \ln(\sec t + \tan t) = \ln\left(1 + \sin\frac{\pi}{2}\right) - \lim_{t \to \frac{\pi}{2}^-} \ln(\cos t) = \infty.$$

The integral is divergent.

Test 3. 1.

(a) Use $u = \ln x$, $du = \frac{dx}{x}$

$$\int \frac{\ln x}{x(\ln x + 1)} dx = \int \frac{u}{1+u} du = \int \left(1 - \frac{1}{u+1}\right) du = u - \ln|1+u| + C = \ln x - \ln|1 + \ln x| + C.$$

(b) Use $u = x + 1$

$$\int \frac{x^2}{(x+1)^{2008}} dx = \int \frac{(u-1)^2}{u^{2008}} du =$$

$$\int \left(\frac{1}{u^{2008}} - \frac{2}{u^{2007}} + \frac{1}{u^{2006}}\right) du = -\frac{1}{2007(1+x)^{2007}} + \frac{2}{2006(1+x)^{2006}} - \frac{1}{2005(1+x)^{2005}} + C$$

(**c**) The substitution $u = \tan x$, $x =$arc $\tan u$, $dx = \frac{du}{1+u^2}$

$$\int \frac{1 - \tan x}{1 + \tan x} dx = \int \frac{1-u}{(1+u)(1+u^2)} du = \int \frac{1}{1+u} - \frac{u}{1+u^2} du =$$

$$\ln|1+u| - \frac{1}{2}\ln\left(1+u^2\right) + C = \ln|1 + \tan x| - \frac{1}{2}\ln|\sec^2 x| + C =$$

$$\ln|1 + \tan x| + \ln|\cos x| + C$$

2.

$$\int e^{2x} \sin x\, dx = -e^{2x}\cos x + 2\int e^{2x}\cos x\, dx,$$

$$2\int e^{2x}\cos x\, dx = 2e^{2x}\sin x - 4\int e^{2x}\sin x\, dx.$$

Therefore

$$\int e^{2x}\sin x\, dx = -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x\, dx,$$

$$\int e^{2x}\sin x dx = \frac{2e^{2x}\sin x - e^{2x}\cos x}{5},$$

$$\int_0^{\pi} e^{2x}\sin x\, dx = \frac{2e^{2x}\sin x - e^{2x}\cos x}{5}\Big|_0^{\pi} = \frac{1 + e^{2\pi}}{5}.$$

3.

$$|\left(\cos x^2\right)''| = |-4x^2\cos x^2 - 2\sin x^2| \leq 6.$$

$$|E_M| \leq \frac{6}{24 \times 100^2} = \frac{1}{4 \times 10^4}.$$

4.

$$\int_0^\infty \frac{dx}{x^2+4x+3} = \lim_{t\to\infty}\int_0^t \frac{1}{(x+3)(x+1)}dx =$$

$$\lim_{t\to\infty}\frac{1}{2}\int_0^t \left(\frac{1}{x+1} - \frac{1}{x+3}\right) dx =$$

$$\frac{1}{2}\lim_{t\to\infty} ln(\frac{x+1}{x+3}) - \frac{1}{2}\ln\frac{1}{3} = \frac{1}{2}\ln\ 3.$$

To calculate $\lim_{t\to\infty} ln(\frac{x+1}{x+3})$ use

$$\lim_{t\to\infty} ln(\frac{x+1}{x+3}) = ln(\lim_{t\to\infty}\frac{x+1}{x+3}) = ln1 = 0.$$