Solutions for sample final

1. Use the formula

$$A = \int_0^1 2\pi\, y ds.$$

We have

$$y = \sqrt{1+e^x},\ \ 0 \le x \le 1,$$

$$ds = \sqrt{1+(y')^2} = \left(\sqrt{\frac{e^{2x}}{4(1+e^x)}+1}\right) dx,$$

$$yds = \sqrt{1+e^x}\left(\sqrt{\frac{e^{2x}}{4(1+e^x)}+1}\right) dx = \frac{1}{2}\sqrt{e^{2x}+4+4e^x}dx,$$

$$A = \pi\int_0^1 \sqrt{e^{2x}+4+4e^x}dx = \pi\int_0^1 (e^x+2)dx = \pi(e+1).$$

2. Take the indefinite integral using the substitution $u = e^x$

$$\int \frac{dx}{e^x+e^{-x}} = \int \frac{du}{u^2+1} = \arctan u + C = \arctan e^x + C.$$

$$\int_0^\infty \frac{dx}{e^x+e^{-x}} = \lim_{t\to\infty}\int_0^t \frac{dx}{e^x+e^{-x}} = \lim_{t\to\infty} \arctan e^t - \arctan 1 = \frac{\pi}{4}.$$

3.

$$\lim_{n\to\infty} \ln(n^{\frac{1}{n}}) = \lim_{n\to\infty}\frac{\ln n}{n} = \lim_{x\to\infty}\frac{\ln x}{x} == \lim_{x\to\infty}\frac{1}{x} = 0.$$

Therefore

$$\lim_{n\to\infty} n^{\frac{1}{n}} = e^{\lim_{n\to\infty}\ln(n^{\frac{1}{n}})} = e^0 = 1.$$

4.The series is absolutely convergent by comparison test. Indeed,

$$|\sin n| \le 1.$$

Therefore

$$|\frac{\sin n}{n^2}| \le \frac{1}{n^2}.$$

Since

$$\sum_{n=1}^\infty \frac{1}{n^2}$$

is convergent,

$$\sum_{n=1}^\infty |\frac{\sin n}{n^2}|.$$

is also convergent.

5. We start with applying the ratio test

$$\lim_{n\to\infty} \frac{\frac{|x|^{n+1}}{\ln(n+1)}}{\frac{|x|^n}{\ln n}} = |x| \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = |x| \lim_{t\to\infty} \frac{\ln t}{\ln(t+1)} = |x| \lim_{t\to\infty} \frac{t+1}{t} = |x|.$$

Hence the series is convergent for $|x| < 1$ and the radius of convergence is 1. To check endpoints, note that for $x = 1$ the series $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ is divergent by the integral test and $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is convergent by the alternating series test. Hence the interval of convergence is $[-1, 1)$.

6. By binomial series

$$\frac{1}{\sqrt{1+u}} = (1+u)^{-1/2} = 1 - \frac{1}{2}u + \frac{3}{8}u^2 + \dots.$$

Substitute $u = -x^2$ and get

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \dots.$$

Integrate the series term by term and use $\arcsin 0 = 0$ to get

$$(\arcsin x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \dots.$$

Therefore

$$T_5(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5.$$

7. Separate the variables

$$(y+1)dy = xdx.$$

After integrating we get

$$\frac{1}{2}y^2 + y = \frac{1}{2}x^2 + C.$$

Use the initial condition $y(0) = 0$ to find that $C = 0$. Therefore the solution of our problem is given by equation

$$y^2 + 2y = x^2$$

or

$$(y+1)^2 = x^2 + 1$$

Can be solved for y

$$y = \sqrt{x^2+1} - 1.$$

8. The integrating factor is $I(x) = e^x$. Multiply the equation by $I(x)$ to get

$$(ye^x)' + 1 = 0.$$

That gives

$$(ye^x)' = -1$$

or

$$ye^x = -x + C.$$

Therefore the general solution is

$$y = -xe^{-x} + Ce^{-x}.$$

9. The auxiliary equation

$$r^2 - 2r + 5 = 0$$

has two complex roots $r_1 = 1 + 2i$ and $r_2 = 1 - 2i$. The general solution is

$$y = C_1 e^x \cos 2x + C_2 e^x \sin 2x.$$

The condition $y(0) = 0$ implies $C_1 = 0$. The second condition $y'(0) = 3$ gives $C_2 = \frac{3}{2}$. The final answer is

$$y = \frac{3}{2} e^x \sin 2x.$$

10. Start with complementary equation

$$y'' - y = 0.$$

The roots of the auxiliary equation are ± 1. Therefore the complementary solution

$$y_c = C_1 e^x + C_2 e^{-x}.$$

Next look for a particular solution of the equation $y'' - y = x^2$ in the form $y = Ax^2 + Bx + C$. After substituting get

$$2A - (Ax^2 + Bx + C) = x^2.$$

Thus, $A = -1$, $B = 0$ and $C = -2$. The solution is $y_1 = -x^2 - 2$.

Now look for a particular solution of the equation $y'' - y = e^x$ in the form $y = Axe^x$.

$$2Ae^x + Axe^x - Axe^x = e^x,\ 2A = 1,\ A = \frac{1}{2},\ y_2 = \frac{xe^x}{2}.$$

To obtain a particular solution of the differential equation

$$y'' - y = x^2 + e^x$$

we use $y_p = y_1 + y_2$. The general solution is

$$y = -x^2 - 2 + \frac{xe^x}{2} + C_1 e^x + C_2 e^{-x}.$$

11. Solve the initial value problem in power series

$$y'' - 2xy' = 2y,\ y(0) = 0,\ y'(0) = 1.$$

Look for the solution in the form

$$y = \sum_{n=0}^{\infty} c_n x^n.$$

After substituting into the equation get

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - 2 \sum_{n=0}^{\infty} n c_n x^n = 2 \sum_{n=0}^{\infty} c_n x^n.$$

That gives the relation

$$(n+2)(n+1)c_{n+2} = (2n+2)c_n$$

or

$$c_{n+2} = \frac{2c_n}{n+2}.$$

The initial conditions imply $c_0 = 0$ and $c_1 = 1$. Therefore $c_n = 0$ for even n. If $n = 2k+1$ we obtain

$$c_{2k+1} = \frac{2^k}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k+1)}$$

and

$$y = \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k+1)}.$$

12.
(a)

$$e^{\pi i/4} = \cos(\pi/4) + i \sin \pi/4 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i.$$

(b)

$$(1-i)^{10} = (\sqrt{2} e^{-\pi i/4})^{10} = (\sqrt{2})^{10} e^{-10\pi i/4} = -32i.$$

13. We use the equation for damped vibration

$$mx'' + cx' + kx = 0.$$

In our case $m = 2$, $c = 8$ and $k = \frac{5}{0.5} = 10$. So our equation is

$$2x'' + 8x' + 10x = 0,$$

or after dividing by 2

$$x'' + 4x' + 5x = 0.$$

The initial conditions are $x(0) = 0.5, x'(0) = 0$. We solve the auxiliary equation

$$r^2 + 4r + 5 = 0.$$

It has two complex roots

$$r_{1,2} = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i.$$

The general solution is given by

$$x = e^{-2t}(C_1 \cos t + C_2 \sin t).$$

The condition $x(0) = 0.5$ implies $C_1 = 0.5$. Using

$$x' = -2e^{-2t}(C_1 \cos t + C_2 \sin t) + e^{2t}(-C_1 \sin t + C_2 \cos t).$$

If $x'(0) = 0$ we have $C_2 = 1$. The solution is given by the formula

$$x(t) = e^{-2t}(0.5 \cos t + \sin t).$$

14. First we solve the complementary equation

$$y'' + 3y' + 2y = 0.$$

The auxiliary equation $r^2 + 3r + 2 = 0$ has two real roots $r_1 = -1$ and $r_2 = -2$. Therefore

$$y_c = C_1 e^{-x} + C_2 e^{-2x}.$$

We look for solution in the form

$$y = u_1 e^{-x} + u_2 e^{-2x},$$

using the method of variation of parameters. We get two equations

$$u_1' e^{-x} + u_2' e^{-2x} = 0, \ -u_1' e^{-x} - 2u_2' e^{-2x} = \cos(e^x).$$

Solving them we get

$$u_1' = e^x \cos(e^x), \ u_2' = -e^{2x} \cos(e^x).$$

Therefore

$$u_1 = \int e^x \cos(e^x) dx, \ u_2 = -\int e^{2x} \cos(e^x) dx.$$

Both integrals can be taken using the substitution $v = e^x$. We obtain

$$u_1 = \sin(e^x) + C_1, \ u_2 = -e^x \sin(e^x) - \cos(e^x) + C_2.$$

The solution

$$y = e^{-x} \sin(e^x) - e^{-x} \sin(e^x) - e^{-2x} \cos(e^x) + C_1 e^{-x} + C_2 e^{-2x} = -e^{-2x} \cos(e^x) + C_1 e^{-x} + C_2 e^{-2x}.$$