1. A field $\mathbb{F}$ is an ordered field if there is a total order $\leq$ on $\mathbb{F}$ satisfying properties
if $a \leq b$, then $a + c \leq b + c$ for all $a, b, c \in \mathbb{F}$;
if $0 \leq a$ and $0 \leq b$, then $0 \leq ab$ for all $a, b \in \mathbb{F}$.
Show that one can not introduce a total order on $\mathbb{C}$ to make it an ordered filed.

2. Let

$$f(z) = \frac{az + b}{cz + d},$$

where a, b, c, d are complex number satisfying $ad - bc \neq 0$. Show that $f(z)$ defines a continuous bijective map of the Riemann sphere to itself and that the set of all such maps is a group with operation of composition.

3. Let $f(z)$ be as in the previous problem. Show that it maps any line in the complex plane to a line or a circle, and any circle to a line or a circle.

4. Give an example of $f(z) = u(x, y) + iv(x, y)$ such that the partial derivatives u_x, u_y, v_x, v_y are defined everywhere in the complex plane, the Cauchy–Riemann condition holds at $z = 0$ but $f'(0)$ does not exist.

5. Prove the Jordan curve theorem for a simple polygon contour.

6. Prove the Jordan curve theorem in case when a simple closed contour C is the union of a graph of some function $y = f(x)$ such that $f(a) = f(b) = 0$ for $a \leq x \leq b$ and the segment $[a, b]$.

7. Give an example of a (non-smooth) simple closed curve of infinite length.

8. Prove that if $f(z)$ is analytic inside and on a closed contour C, then for any z inside C

$$f^{(n)} = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w - z)^{n+1}} dw.$$

9. Let $f(z)$ be analytic on a domain D and $f'(z) \neq 0$ for any $z \in D$. Prove that $f(D)$ is a domain.

10. Let $f(z)$ be continuous on a domain D and analytic at all except finitely many points in D. Prove that $f(z)$ is analytic on D.

11. Let $f(z)$ be an entire function and $f(z) = 0$ for all real z. Show that $f(z) = 0$ for all z.

12. Let $f(x) = e^{-1/x^2}$ if $x \neq 0$ and $f(0) = 0$. Show that for all $n > 0$ $f^{(n)}(x)$ is continuous on the entire real line. Show that $f(x)$ can not be extended to an entire function on the complex plane.

13. Let $f(z) = \frac{az+b}{cz+d}$, where a, b, c, d are complex number satisfying $ad - bc = 1$. Let $\{z_n\}$ be a sequence defined by $z_{n+1} = f(z_n)$. Find all a, b, c, d and z_1 such that $\lim_{n\to\infty} z_n$ exists.

14. Let a Laurent series

$$\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

converge in an annular domain $D = \{z \in \mathbb{C} | R_1 < |z - z_0| < R_2\}$.

(a) Show that the series uniformly converges in the closed ring $r_1 \le |z - z_0| \le r_2$ for any $R_1 < r_1 < r_2 < R_2$.

(b) Show that the sum

$$S(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

is continuous in D.

(c) Show that for any contour C inside D and a continuous function $g(z)$ on C

$$\int_C g(z)S(z)dz = \sum_{n=-\infty}^{\infty} c_n \int_C g(z)(z - z_0)^n dz.$$

(d) Show that $S(z)$ is analytic in D.

15. (For people who took set theory and linear algebra.) Let V be a vector space of functions analytic at $z = 0$. Show that the dimension of V is not countable.

16. Find a function analytic at 0 with Maclaurin series

$$\sum_{n=0} \binom{2n}{n} z^n.$$

What is the circle of convergence for this series?

17. Use the previous problem to prove the identity

$$\sum_{k=0}^{n} \binom{2(n-k)}{n-k} \binom{2k}{k} = 4^n.$$

18. Let $f(z)$ have an essential singularity at $z = z_0$ and w be a complex number. Show that there exists a sequence $\{z_n\}$ such that

$$\lim_{n\to\infty} z_n = z_0, \quad \lim_{n\to\infty} f(z_n) = w.$$

19. Let z_0 be an isolated singular point. Show that if

$$\lim_{z\to z_0} f(z) = \infty,$$

then z_0 is a pole.

20. We say that $f(z)$ has a pole at ∞ if

$$\lim_{z\to\infty} f(z) = \infty.$$

Prove that if $f(z)$ has finitely many singular points which are all poles including ∞, then $f(z)$ is a rational function.

21. Construct a conformal bijective map from the interior of a rectangle to the interior of the unit disk.

22. Let A and B be two annular domains $r_1 < |z| < R_1$ and $r_2 < |z| < R_2$ and let there exist a bijective conformal mapping $f : A \to B$. Show that $\frac{R_1}{r_1} = \frac{R_2}{r_2}$.