ELLIPTIC FUNCTIONS IN EXERCISES

ILYA ZAKHAREVICH

Abstract. Solving this handful of semi-hard exercises would make you ready to work with elliptic functions.

Contents

0.	Introduction	1
1.	The principal players	2
2.	Normalizations	2
3.	Values at semi-periods	4
4.	Conformal mappings	5
5.	Infinite products	5
6.	Weierstrass ζ-function and representations of doubly-periodic functions	5
7.	Uniformization of cubic curves	7
8.	Elliptic integrals	8
9.	Curves of genus 1	10
10.	Cubic curves	11
11.	Elliptic quartics	14
References		15

0. Introduction

Personally, I find it very hard to learn a new domain unless I discover most key concepts myself. The exercises below mark a path through the marches so that this rediscovery will not take as much time as the original discovery did.

Very few of the exersizes are trivial—unless you know already how to solve them. Very few should require more than a half an hour to solve, thus if it takes more time that this, it may be more useful to consult a book. If difficulties arise, a good book on complex analysis would help; I prefer [1].

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1. The principal players

A two-periodic (or *elliptic*) *function* is a function $f(z)$, $z \in \mathbb{C}$, such that $f(z+T_{1,2}) = f(z)$; here $T_{1,2} \in \mathbb{C}$ are $\mathbb{R}$-linearly independent. If we consider $\mathbb{C}$ as 2-dimensional vector space over $\mathbb{R}$, then $\mathbb{Z}$-linear combinations of T_1, T_2 form a *lattice* L of *periods*. A *fundamental domain* $U \subset \mathbb{C}$ is a closed subset with a piecewise-smooth boundary such that L-translations of U cover $\mathbb{C}$ and intersect only along boundaries. When we count points in fundamental domains, we always assume for simplicity that the points do not meet the boundary.

Label exer10

Exercise 1.1. Show that there exists a 2-periodic meromorphic function $p(z)$ which has exactly one pole on the fundamental domain, and this pole is of order 2.

Exercise 1.2. Show that there is no 2-periodic meromorphic function which has exactly one pole on the fundamental domain, and this pole is of order 1.

Label exer30

Exercise 1.3. Show that $p(z)$ satisfies an ODE of the first order with constant coefficients. (In fact, this ODE is implicit, as in $F(p,p') = 0$.)

Label exer40

Exercise 1.4. Show that $p(z)$ satisfies an ODE of the form

$$p''(z) = F(p(z));$$

here F is a polynomial of degree 2.

Label exer54

Exercise 1.5. Show that there exists a holomorphic function $\vartheta(z)$ such that $p = (\ln \vartheta)''$; here p satisfies the conditions of Exercise 1.1.

2. Normalizations

The functions $p(z)$ and $\vartheta(z)$ of the previous section are not uniquely defined. There are natural ways to choose a "best" representative of such a function. When there are several independent ways, the reduction of one of the ways to another leads to useful relations between elliptic functions.

Exercise 2.1. The function $\vartheta(z)$ of Exersize 1.5 is subject to transformations of the form $\vartheta(z) = ae^{bz+cz^2}\widetilde{\vartheta}(z+d)^n$, $n \in \mathbb{N}$. Show that one can chose $\widetilde{\vartheta}(z)$ in such a way that *any* function ϑ of Exersize 1.5 can be expressed by such a formula.

Functions $\widetilde{\vartheta}$ from the previous exersize are characterized by the condition that they do not have multiple zeros. In what follows, the notation ϑ is used only for functions of such form.

A function $f(z)$ has a λ-*quasiperiod* T if $f(z+T) \equiv \lambda f(z)$.

Exercise 2.2. Show that that $\vartheta(z)$ can be chosen having T_1 as a period. In fact, for any $\lambda \neq 0$, $\vartheta(z)$ can be chosen in such a way that T_1 is a λ-quasiperiod, i.e., $\vartheta(z+T_1) \equiv \lambda\vartheta(z)$. The choice of λ determines ϑ up to a transformation $\vartheta(z) = a\vartheta(z+b)$.

It is convenient to be able to restrict the freedom in the choice of $\vartheta(z)$ yet more. In the previous exercise, ϑ is determined by the lattice L of periods together with a subgroup generated by T_1. These data is invariant w.r.t. reflection $z \mapsto -z$.

Label exer50

Exercise 2.3. Show that $\vartheta(z)$ can be chosen so that $\vartheta(z)$ is either even, or odd. How many such choices of $\vartheta(z)$ exist (up to a multiplicative constant)? Same questions under the condition that $\vartheta(z)$ has 1 as a λ-quasiperiod.

The previous exersize shows that one can significantly restrict the freedom in the choice of $\vartheta(z)$, as well as importance of $\lambda = \pm 1$.

To simplify normalizations, suppose that $T_1 = 1$, $T_2 = \tau$. Since $\operatorname{Im}\tau$ cannot be 0, it is convenient to restrict the attention to the connected component $\operatorname{Im}\tau > 0$ (the other case is similar due to invariance w.r.t. $\tau \mapsto -\tau$). Suppose also that $\vartheta(z)$ has 1 as a ± 1-quasiperiod, and is either even, or odd.

Label exer60

Exercise 2.4. Express $\vartheta(z+1)$, $\vartheta(z+\tau)$ in terms of $\vartheta(z)$ for all possible choices of such functions $\vartheta(z)$.

Exercise 2.5. Find Fourier coefficients for $\vartheta(x)$, $x \in \mathbb{R}$, for all possible choices of such functions $\vartheta(z)$. *Hint:* for some choices the period is 1, for some it is 2.

Label exer80

Exercise 2.6. The conditions above determine $\vartheta(z)$ up to a multiplicative constant and a discrete parameter. This defines several functions of *two* complex variables $\vartheta(z,\tau)$; they are defined up to a multiplication by a function of τ and a discrete parameter.

Show that there is a choice of $\vartheta(z,\tau)$ in the region $\operatorname{Im}\tau > 0$ which satisfies a PDE $C\frac{\partial\vartheta}{\partial\tau} = \frac{\partial^2\vartheta}{\partial z^2}$ for an appropriate choice[1] of a constant C.

Exercise 2.3 gives a list of several "good" normalizations for the function ϑ. It is convenient to denote these choices by the positions of zeros of ϑ. Chose the parallelogram built on $T_{1,2}$ as a fundamental domain; then denote[2] an even-or-odd functions ϑ depending on the position of the zero inside this parallelogram: the function which has a zero at $\alpha T_1/2 + \beta T_2/2$ is $\vartheta_{\alpha,\beta}$. Obviously, $\alpha, \beta \in \{0,1\}$. These functions of z are determined up to multiplication by a constant.

On the other hand, Exercise 2.6 provides a way to normalize the functions $\vartheta_{\alpha,\beta}(z)$ up to a multiplicative constant which *does not depend on* τ. The convenient choice of this multiplicative constant is given by the following exercises:

Exercise 2.7. Find the quasiperiod-factors λ for $\vartheta_{\alpha,\beta}(x)$.

Exercise 2.8. The average value of $\vartheta_{\alpha,1}(x)$, $x \in \mathbb{R}$, on its period does not vanish. In the conditions of Exercise 2.6 the average value does not depend on τ.

[1] This is the first time we used the condition $\operatorname{Im}\tau > 0$.

[2] This notation is slightly different from the Jacobi notation $\Theta_{\alpha,\beta}$; the latter functions are defined for all integer α and β, and differ from our notation by a shift of α and β by 1.

Normalize $\vartheta_{\alpha,1}(z)$ by requiring that the average value above is 1. This makes $\vartheta_{\alpha,1}$ into a well-defined function depending on z and τ. The notation $\vartheta_{\alpha,1}(z)$ may be used when τ is assumed to be a fixed number.

With the normalization above, $\vartheta_{\alpha,1}$ satisfies the equation of Exercise 2.6. Similarly, it is convenient to normalize $\vartheta_{\alpha,0}$ applying the equation of Exercise 2.6 to the first Fourier coefficient $a_1(\tau)$ (instead of the 0-th one—proportional to the average value—above). By Exercise 2.6, $a_1(\tau) = Ae^{-B\tau}$ with a known constant B (here $\operatorname{Im}\tau > 0$). The standard normalization makes the first Fourier term of $\vartheta_{\alpha,0}(z)$ at $\tau = i$ into $e^{-\pi/8}e^{\pi i z}$.

Note that the function $p(z)$ is automatically even if its pole is at 0.

Exercise 2.9. Normalize $p(z)$ so that it has a pole at 0 with the leading term of Laurent series z^{-2}. Express $p(z)$ in terms of $\vartheta_{\alpha,\beta}$ for all choices of $\alpha, \beta \in \{0,1\}$.

Exercise 2.10. Show that in the conditions of the previous exercise $p(z) - p(1/2)$ has a zero of order 2 at $1/2$; moreover, this difference (as a function of z) is uniquely determined by τ.

Exercise 2.11. Express $p(\tau/2) - p(1/2)$ in terms of three numbers $\vartheta_{\alpha,\beta}(0)$, $\alpha, \beta \in \{0,1\}$ (three since $\vartheta_{0,0}(0) \stackrel{\text{def}}{=} 0$).

Exercise 2.12. Show that there exists a number C such that $p(\tau/2) - p(1/2) = C\vartheta''_{1,1}(0)$. Find C.

The conditions on the pole (position and the leading coefficient) determine $p(z)$ up to an addititive constant. Chose this constant to kill the 0-th Laurent coefficient near 0; the resulting function is called the Weierstrass $\mathcal{P}$-function. It depends on z and τ, but usually we write $\mathcal{P}(z)$ assuming τ fixed.

Remark 2.13. Note that this normalization of p and the normalization of $\vartheta_{0,0}$ above is not compatible with Exercise 1.5 (see Exercise 5.2).

To define ϑ-function for an arbitrary lattice $\langle T_1, T_2\rangle$, transfer it from the lattice $\langle 1, T_2/T_1\rangle$ by multiplication: $\vartheta(z|T_1,T_2) \stackrel{\text{def}}{=} \vartheta(z/T_1|1, T_2/T_1)$. The normalization of the $\mathcal{P}$-function defined above is applicable for an arbitrary lattice of periods. However, if not stated otherwise, the lattice is assumed to be with $T_1 = 1$ if τ is mentioned in the context.

3. Values at semi-periods

Exercise 3.1. Show that $\mathcal{P}(1/2) = \text{const}\,\frac{d}{d\tau}\log\vartheta_{1,0}(0)$; both sides are functions of τ.

Exercise 3.2. Write similar formulae for $\mathcal{P}(\tau/2)$ and $\mathcal{P}((1+\tau)/2)$.

The usual notations for the values $\mathcal{P}(T_1/2)$, $\mathcal{P}((T_1+T_2)/2)$, and $\mathcal{P}(T_2/2)$ are e_1, e_2, e_3. The ordering corresponds to a counter-clockwise walk around the parallelogram with $T_1/2$, $T_2/2$ as sides adjacent to the vertex 0.

Exercise 3.3. Find periods of the meromorphic functions $\sqrt{\mathcal{P}(z)-\mathcal{P}(e_k)}$, $k = 1, 2, 3$.

4. Conformal mappings

Exercise 4.1. If $\tau \in i\mathbb{R}_{>0}$, then the function $\vartheta_{\alpha,\beta}(z)$, $z \in \mathbb{R}$, are real. Moreover, $\mathcal{P}(z)$ and $\mathcal{P}(z+\tau/2)$, $z \in \mathbb{R}$, are real.

Exercise 4.2. If $\tau \in i\mathbb{R}_{>0}$, then the correspondence $t = -\mathcal{P}(z)$ is a conformal mapping of the (rectangular) parallelogram above to $\{\operatorname{Im} t > 0\}$.

Exercise 4.3. If $\tau = i$, then the correspondence $t = \frac{\mathcal{P}(z)+ie_1}{\mathcal{P}(z)-ie_1}$ is a conformal mapping of the (square) parallelogram above to the unit circle which commutes with $\pi/4$-rotations of the square and the circle.

Label exer120

Exercise 4.4. One has $e_1 + e_2 + e_3 = 0$. If $\tau \in i\mathbb{R}_{>0}$, then $e_3 < e_2 < e_1$.

To restore the symmetry between e_k and T_k, one can define *half-periods* $\omega_{1,2,3}$ as $\omega_1 = T_1/2$, $\omega_2 = -(T_1 + T_2)/2$, $\omega_3 = T_2/2$. Then $\omega_1 + \omega_2 + \omega_3 = 0$ and $\mathcal{P}(\omega_k) = e_k$.

Exercise 4.5. Find a conformal mapping of a given ellipse to a circle. *Hint:* the Zhukovsky's mapping $t = z + 1/z$ sends the annulus $\{1 < |z| < a\}$ to an ellipse with a removed interval on the major axis.

5. Infinite products

Exercise 5.1. Write a representation of $\vartheta_{\alpha,\beta}(z)$ as an infinite product over its zeros.

Label exer200

Exercise 5.2. Show that $\mathcal{P} = -\left(\frac{\vartheta'_{0,0}}{\vartheta_{0,0}}\right)' + \frac{1}{3}\frac{\vartheta'''_{0,0}(0)}{\vartheta'_{0,0}(0)}$.

6. Weierstrass ζ-function and representations of doubly-periodic functions

The antiderivative of the function $\mathcal{P}(z)$ exists as a merormorphic function. It is uniquely determined by the condition that it is odd; denote this function as $-\zeta(z)$. The principal part of the function $\zeta(z)$ near $z = 0$ is $1/z$.

This function is called Weierstrass ζ-function. Do not mix it with Riemann ζ-function!

Use Exercises 2.4, 5.2 for the following:

Exercise 6.1. The function $\zeta(z)$ is not 2-periodic. Denote $\zeta(z+T_k) - \zeta(z)$ by $2\eta_k \in \mathbb{C}$, $k = 1, 2$. Then $\eta_1 = -\frac{T_1}{6}\frac{\vartheta'''_{0,0}(0)}{\vartheta'_{0,0}(0)}$; $\eta_2 = -\frac{T_2}{6}\frac{\vartheta'''_{0,0}(0)}{\vartheta'_{0,0}(0)} - \pi i/T_1$.

In particular, the formulae above imply the *Legendre relation*

$$\eta_1 T_2 - \eta_2 T_1 = \pi i.$$

Exercise 6.2. Prove this relation independently. If $T_1 = 1$, $T_2 = i$ (*lemniscatic case*) this implies $\eta_1 = \pi/2$; similarly, if $T_1 = 1$, $T_2 = \frac{1+i\sqrt{3}}{2}$, then $\eta_1 = \pi/\sqrt{3}$. *Hint: use the residue formula.*

Alternatively, η_k can be defined as $\zeta(T_k/2)$.

Exercise 6.3. Show that $\zeta(\omega_2) = -\eta_1 - \eta_2$.

Exercise 6.4. If $\tau \in i\mathbb{R}_{>0}$, then the correspondence $t = \zeta(z) + e_2 z$ is a conformal mapping of the (rectangular) parallelogram fundamental domain centered at 0 to the complex sphere with a removed rectangle with vertices at $\pm a \pm$bi; here $a = \eta_1 + e_2 T_1/2$, $ib = \eta_2 + e_2 T_2/2$.

Exercise 6.5. If $\tau \in i\mathbb{R}_{>0}$, then the correspondence $t = \zeta(z) - \frac{2\eta_2}{T_2} z$ is a conformal mapping of the (rectangular) parallelogram fundamental domain centered at 0 to the complex sphere with cuts along $[-a, a]$, $[-a - ib, -a + ib]$, and $[a - ib, a + ib]$; here $a = \pi i/T_2$, b is an appropriate number.

Exercise 6.6. There exists τ which corresponds to $\eta_2 = 0$.

In particular, ζ is not necessary a 1-to-1 mapping from the fundamental parallelogram centered at 0.

Exercise 6.7. Consider an arbitrary meromorphic elliptic function $f(z)$ with simple poles $z_1, \dots, z_K$ on the fundamental domain with residues $a_1, \dots, a_K$. Then $f(z) = \sum_{k=1}^{K} a_k \zeta(z - z_k) + \text{const}$.

In particular, T_1-periodicity of the last expression implies that $\sum a_k = 0$ (this is easy to prove independently). Similarly, one can allow higher orders of poles by allowing additional terms of the form $a_{k,l}\zeta^{(l)}(z - z_k)$.

Label exer165

Exercise 6.8. Consider an arbitrary meromorphic elliptic function $f(z)$. There is a meromorphic elliptic function $g(z)$ and numbers K, $a_k, z_k \in \mathbb{C}$, $k = 1, \dots, K$, C_1, C_2 such that $f(z) - g'(z) = \sum_{k=1}^{K} a_k \zeta(z - z_k) + C_1 \mathcal{P}(z) + C_2$.

Label exer170

Exercise 6.9. Consider an arbitrary 2-periodic meromorphic function $f(z)$ with zeros and poles on the fundamental domain at points $z_1, \dots, z_K$ with multiplicities n_k (write n_k positive for zeros, negative for poles). Then $f(z) = \text{const} \prod_{k=1}^{K} \vartheta_{0,0}(z - z_k)^{n_k}$.

In particular, T_2-periodicity of the last expression implies that $\sum n_k = 0$, and that $\sum n_k z_k$ is in the lattice.

Exercise 6.10. Prove these relations independently. *Hint: use the residue formula.*

Label exer6.80

Exercise 6.11. Show that if f is a meromorphic function, and f' is 2-periodic, then $f(z) = \sum_{k=1}^{K} a_k \zeta^{(l_k)}(z - z_k) + C_1 z + C_2$, for an appropriate choice of C_1, C_2, K, a_k, z_k, and l_k.

Label exer6.90

Exercise 6.12. Show that if $f(z)$ is a meromorphic function, and $(\log f)''$ is 2-periodic, then $f(z) = C_1 e^{C_2 z + C_3 z^2} \prod_{k=1}^{K} \vartheta_{0,0}(z - z_k)^{a_k}$, for an appropriate choice of C_1, C_2, C_3, K, z_k, and a_k.

7. Uniformization of cubic curves

Now we have more notations and normalizations to make the Exercise 1.3 more precise.

Recall that the projective plane $\mathbb{CP}^2$ consists of lines in $\mathbb{C}^3$ passing through the origin; the line through $(X, Y, Z) \in \mathbb{C}^3$ is denoted as $(X : Y : Z)$. The affine plane $\mathbb{C}^2$ is a subset of $\mathbb{CP}^2$ via $(X, Y) \to (X : Y : 1)$; the complement is a projective line $l_\infty \simeq \mathbb{CP}^1$; it consists of points of the form $(X : Y : 0)$.

Label exer7.10

Exercise 7.1. Fix two numbers g_4, g_6. Denote by E_0 the set of solutions of $Y^2 = 4X^3 - g_4 X - g_6$; here $(X, Y) \in \mathbb{C}^2$. Consider $\mathbb{C}^2$ as a subset of the projective plane $\mathbb{CP}^2$; let E be the closure of E_0 in $\mathbb{CP}^2$. Then the closure E of E_0 in $\mathbb{CP}^2$ is $E_0 \cup \{P_\infty\}$; here $P_\infty = (0 : 1 : 0)$.

The curve E is called the *projective closure* of E_0.

In addition to $(X : Y : 1)$, one can consider other coordinate charts on $\mathbb{CP}^2$. The chart $(X : Y : 1)$ works near $(X_0 : Y_0 : Z_0)$ iff $Z_0 \neq 0$; similarly, the chart $(X : 1 : Z)$ works iff $Y_0 \neq 0$, and the chart $(1 : Y : Z)$ works iff $X_0 \neq 0$. Since transition mappings between these charts are complex analytic, $\mathbb{CP}^2$ is a complex analytic manifold, thus it makes sense to consider complex analytic submanifolds of $\mathbb{CP}^2$.

Exercise 7.2. The neighborhood of P_∞ on E is complex analytic; $t = 2X/Y$ is a coordinate system near P_∞. In this coordinate system $X - t^{-2}$ and $Y - 2t^{-3}$ (defined for small $t \neq 0$) can be extended to $t = 0$ as complex analytic functions of t.

The tangent line to E at P_∞ coincides with l_∞. In fact, l_∞ has a tangency of the second order with E; in other words, P_∞ is an inflection point on E.

Exercise 7.3. Let $\Delta \overset{\text{def}}{=} \Delta_{12} = g_4^3 - 27g_6^2$. The manifold E is complex analitic iff $\Delta \neq 0$. This happens iff $4X^3 - g_4 X - g_6$ has 3 distinct roots.

Exercise 7.4. If $\Delta \neq 0$, then E is homeomorphic to a torus; in other words, E is a compact orientable surface of Euler characteristic 0.

Recall that *a complex curve of genus g* is a 1-dimensional complex analytic manifold homeomorphic to a compact surface of Euler characteristic $2 - 2g$ (i.e., to a sphere with g handles). It is automatically orientable. Thus E is complex curve of genus 1.

Label exer205

Exercise 7.5. Let L be the lattice of periods generated by T_1 and T_2. Let $e_{1,2,3}$ be the corresponding values of the $\mathcal{P}$-function at half-periods, and $4(X - e_1)(X - e_2)(X - e_3) = 4X^3 - g_4 X - g_6$ (compare with Exercise 4.4). The mapping $U \colon z \mapsto (\mathcal{P}(z), \mathcal{P}'(z)) \in \mathbb{C}^2$ is well-defined on $\mathbb{C} \smallsetminus L$; extend it to a mapping $\mathbb{C} \to \mathbb{CP}^2$ by $U(z) = P_\infty$ if $z \in L$. Then U send $\mathbb{C}$ onto E; $U(z) = U(z')$ iff $z' - z \in L$.

In other words, U identifies the quotient space $\mathbb{C}/L$ with E. (Such an identification is called *uniformization.*) In particular, any elliptic function with the lattice of periods L can be considered as a function on E. Because of this, E is called an *elliptic curve.*

Note that the domain $\mathbb{C}$ of the variable z is naturally identified with the *universal cover* $\bar{E}$ of the torus E.

Exercise 7.6. If an entire analytic function is a bijection $\mathbb{C} \to \mathbb{C}$, it coincides with $az + b$.

Thus the automorphisms of $\mathbb{C}$ have the form $\widetilde{z} = az + b$, and the identification of $\bar{E}$ with $\mathbb{C}$ is defined uniquely up to such an automorphism. In the settings above we normalized the variable a by requiring that dz goes to dX/Y, and normalized b by the requirement that 0 maps to $P_\infty \in E$.

8. Elliptic integrals

The functions X, Y on $\mathbb{C}^2$ cannot be extended to $\mathbb{CP}^2$ as a holomorphic function; however, they can be extended as *meromorphic functions*; in other words, in any chart on $\mathbb{CP}^2$ they can be written as f/g; here f and g are holomorphic functions. Thus the differential forms dX and dY on $\mathbb{C}^2$ can be extended as meromorphic differential forms (i.e., forms which can be written as ω/g, ω being a holomorphic differential form, g being a holomorphic function) on $\mathbb{CP}^2$.

Exercise 8.1. The differential forms dX and dY on $\mathbb{C}^2$ have a pole of the second order on l_∞; in other words, if $u = 0$ is the equation of l_∞ in a coordinate chart on $\mathbb{CP}^2$, then $u^2 dX$ and $u^2 dY$ are holomorphic differential forms on this chart.

Exercise 8.2. Consider the elliptic curve E from the previous section. When restricted to E, the meromorphic differential forms dX and dY on $\mathbb{C}^2$ have poles of the 3rd and the 4th order correspondingly. In other words, if $t = 0$ is the equation of P_∞ in a coordinate chart on E, then $t^3 dX$ and $t^4 dY$ are holomorphic differential forms on this chart.

Label exer300

Exercise 8.3. Consider two rational functions R and Q of two variables. Consider the restriction ω_{RQ} of the differential form $R(X,Y)\,dX + Q(X,Y)\,dY$ to E. Assume that $\omega_{RQ} \neq 0$. Then ω_{RQ} is holomorphic on E iff E is complex analytic and ω_{RQ} is proportional to dX/Y. Any such holomorphic form has no zeros on E.

Exercise 8.4. Consider the mapping U from Exercise 7.5, and the differential form $\omega = dX/Y$ from Exercise 8.3. Then $U^*\omega = dz$.

Given a holomorphic differential 1-form ω, the *set of periods* of this form consists of numbers $p \in \mathbb{C}$ which can be written as $\int_C \omega$ for a closed curve C. Since these numbers do not change when one changes the curve C in a continuous family C_s, they depend only on the topological class of C. Clearly, they form a group w.r.t. addition.

Label exer340

Exercise 8.5. Consider the differential form $\omega = dX/Y$ from Exercise 8.3. Suppose that E is complex-analytic. Then there are numbers $T_1, T_2 \in \mathbb{C}$ such that the set of periods of ω coincides with the $\mathbb{Z}$-lattice L generated by T_1, T_2.

Label exer350

Exercise 8.6. Consider the differential form $\omega = dX/Y$ from Exercise 8.3 and the lattice L from Exercise 8.5. Then for any point $p \in E$ there is a number $z \in \mathbb{C}$ such that $z - \int_C \omega \in L$ for any curve C on E going from p_∞ to p. Number z is defined uniquely by p up to addition of the element of L.

Label exer360

Exercise 8.7. The values e_1, e_2, e_3 corresponding to the lattice L of Exercisse 8.5 coincide with the roots of the polynomial $4X^3 - g_4X - g_6$. Thus any cubic curve of the form $Y^2 = 4X^3 - g_4X - g_6$ can be uniformized as in Exercise 7.5.

Exercise 8.8. Consider a meromorphic differential form ω on E. Then there are rational functions R and Q (as in Exercise 8.3) such that $\omega = \omega_{RQ}$. In fact one can take $Q = 0$. *Hint: one can kill the poles on E_0 by multiplying by $X - X_0$ for an appropriate X_0. One can kill the remaining poles at P_∞ by subtracting X^n or X^nY. (One can also apply uniformization and Exercise* **?** exer170.)**?** exer170.) exer170.)

In other words, consideration of meromorphic differential forms on E is equivalent to consideration of integrals of the form $\int R\left(X, \sqrt{4X^3 - g_4X - g_6}\right) dX$; here $R(X, Y)$ is a ratio of two polynomials. Such integrals are called *elliptic integrals*. We already saw that the elliptic integral for $R(X, Y) = 1/Y$ solves the problem of uniformization: find a lattice L such that E can be uniformized by $\mathbb{C}/L$.

Exercise 8.9. Write the integral for the length of the arc of an ellipse as an elliptic integral.

In fact it is this problem which lead to the word *elliptic* for elliptic integrals, functions, and curves. Measuring the length of the arc counterclockwise, one obtains a differential 1-form $\omega = ds$ on the ellipse.

Label exer400

Exercise 8.10. The above discussion defines ω on the real part of the ellipse only. This form has no analytic extension to the complex ellipse C (i.e., the set of complex solutions (x, y) of the equation $x^2/a^2 + y^2/b^2 = 1$). However, it has a "two-valued" extension $\pm\omega$ to C; in other words, there is a (ramified) 2-sheet covering E of C and an analytic 1-form ω_E on E which coincides with ω or $-\omega$ on the preimage of the real part of the ellipse.

Exercises of this section imply that any elliptic integral can be written as an integral of $\omega = f\,dX/Y$; here $f(X, Y)$ is a meromorphic function on the elliptic curve. Using uniformization coordinate z and Exercise 6.8, one can reduce this integral to the cases $f = 1$ (when $\omega = dz$), $f = X$ (when $\omega = d\zeta$; this corresponds to the term $\mathcal{P}(z)$ in Exercise 6.8), and $f(X, Y) = \frac{Y+Y_0}{2X-2X_0}$ with $(X_0, Y_0) \in E$ (corresponding to the term $f_{z_0}(z) \overset{\text{def}}{=} \zeta(z - z_0) - \zeta(z)$ in Exercise 6.8).

Exercise 8.11. Show that $f_{z_0}(z) = \frac{Y+Y_0}{2X-2X_0}$.

It is clear that $\exp\int f_{z_0}(z)\,dz$ is a meromorphic function; by Exercise 6.12 it can be written as $C_1 e^{C_2 z}\vartheta_{00}(z-z_0)/\vartheta_{00}(z)$. Consequently, one can write a formula for an arbitrary elliptic integral written as a function of z in terms of z, $\mathcal{P}(z)$ (and its derivatives), $\zeta(z)$, and $\log\vartheta(z-z_k)$; here z_k are the values of z at the poles of the expression under the integral. Moreover, this procedure is completely algorithmic—except for the value of the constant C_2 above.

One can write the formula for the constant C_2, however, it is more convenient to normalize the function f_{z_0} differently. Using $d\log(\mathcal{P}(z)-\mathcal{P}(z_0))/dz = \zeta(z-z_0)+\zeta(z+z_0)-2\zeta(z)$, one can reduce integration of f_{z_0} to integration of $g_{z_0} = \zeta(z-z_0)-\zeta(z+z_0) = C/(\mathcal{P}(z)-\mathcal{P}(z_0))$, which corresponds to so called *elliptic integral of the third kind* $\int\frac{dX}{(X-c)Y}$. Since g_{z_0} is an odd function, $\exp\int g_{z_0}(z)\,dz$ is $C_1 e^{C_2 z}\vartheta_{00}(z-z_0)/\vartheta_{00}(z+z_0)$.

Exercise 8.12. Show that $C_2 = 2\zeta(z_0) + \frac{2}{3}\vartheta_{00}'''(0)/\vartheta_{0,0}'(0)$.

By Exercise 1.3, derivatives of $\mathcal{P}$ may be written as functions of $X = \mathcal{P}$ and $Y = \mathcal{P}'$, one can conclude that elliptic integrals can be written as a sum of rational function $R(X,Y)$, of several terms proportional to $\log(X-X_0)$, and of terms proportional to z, $\zeta(z)$, and $\log\frac{\vartheta_{00}(z-z_0)}{\vartheta_{00}(z+z_0)}$. The coefficients for the terms $R(X,Y)$, $\zeta(z)$ and $\log\frac{\vartheta_{00}(z-z_0)}{\vartheta_{00}(z+z_0)}$ can be explicitly calculated given *the singular part*[3] of the meromorphic differential form $f(X,Y)\,dX/Y = \widetilde{f}(z)\,dz$ near its poles. The coefficient at z can be calculated given the 0-th Laurent coefficient of $\widetilde{f}$ at 0. (One does not need any other information about f since the singular parts and the 0-th Laurent coefficient at a point determine f uniquely.)

When one considers the elliptic integrals as (multivalued) functions of X, the function z is called *the integral of the first kind*, the function $\zeta(z)$ is *the integral of the second kind*.

9. Curves of genus 1

As defined above, a curve of genus 1 is a 1-dimensional complex analytic manifold which is a torus when considered as a real surface. The target of this section is to show that any abstract elliptic curve is isomorphic to an elliptic curve.

A *ramified covering* $C \xrightarrow{\pi} C'$ is a mapping of complex curves which is locally invertible on a complement to a finite subset of C. A ramified covering is an *n-sheet covering* if the preimage of a point $P \in C'$ consists of n points if P is in the complement to a finite subset $R \subset C'$; one says that π is *ramified over* R.

Label exer445

Exercise 9.1. Consider a complex analytic manifold C which is isomorphic to $\mathbb{CP}^1$; consider its ramified 2-sheet covering E ramified over $\{x_1,\dots,x_N=\infty\} \subset C$.

[3]Negative Laurent coefficients.

Then E is isomorphic to the projective closure of the set of solutions (x, y) to $y^2 = \prod_{n=1}^{N-1} (x - x_N)$.

Exercise 9.2. Show that the curve E from Exercise 8.10 is isomorphic to an elliptic curve. This isomorphism sends the differential form ds to a form proportional to dX/Y.

Given a 1-dimensional complex analytic manifold C and a point $P \in C$, define Γ_k as the vector space of complex analytic functions on $C \setminus \{P\}$ which have at most a pole of order k at P. Let $\gamma_k = \dim \Gamma_k$. Recall that any function $f \in \Gamma_k$ gives a complex analytic mapping $C \to \mathbb{CP}^1$.

Exercise 9.3. $\gamma_{k+l} \leq \gamma_k + l$ if $l \geq 0$. Suppose that C is compact. Then $\gamma_0 = 1$. If $f \in \Gamma_k \setminus \Gamma_{k-1}$, then f gives a k-sheet ramified covering $C \to \mathbb{CP}^1$. Thus if $\gamma_1 = 2$, and $f \in \Gamma_1 \setminus \Gamma_0$, then $f \colon C \to \mathbb{CP}^1$ is an isomorphism; hence $\gamma_k = k + 1$ if $k \geq 0$. *Hint: use inverse function theorem to appropriate reparametrization of f.*

Exercise 9.4. Suppose that $\gamma_2 = 2$. Then C is isomorphic to a 2-sheet covering of $\mathbb{CP}^1$ ramified at $2g + 2$ points; here g is the genus of C.

The curves of the previous exercise are called *hyper-elliptic curves*; their structure is described by Exercise 9.1. In particular, any curve of genus 1 with $\gamma_2 \geq 2$ is an elliptic curve (in particular, $\gamma_k = k$ for $k \geq 2$).

In fact, any curve of genus 1 has $\gamma_2 = 2$; thus it is an elliptic curve. However, to prove this one needs to show the existence of a non-constant meromorphic function with the only pole of order 2 at P. This is a not-trivial problem: essentially, one needs to show that the corresponding Cauchy–Riemann equation has a global solution; this is more or less equivalent[4] to a calculation of an index of a differential operator (*index formula* or *Riemann–Roch formula*).

In fact, one can make the previous Exercise more precise:

Exercise 9.5. Suppose that $\gamma_3 = 3$. Then C is of genus 1. *Hint: denote by f_2 and f_3 functions with poles of the corresponding order; then one can construct 7 monomials of $f_{2,3}$ in Γ_6.*

10. Cubic curves

Consider a cubic polynomial $Q_3(X, Y)$; let $E_0 \subset \mathbb{C}^2$ be a *cubic curve*; i.e., it consists of solutions to $Q_3 = 0$. Let E be the projective closure of E_0 (see Exercise 7.1); it is also called a *cubic curve*. Assume that E contains no projective line.

Exercise 10.1. E contains a projective line iff Q_3 can be written as a non-trivial product.

[4] Is it possible to reduce this particular case to the fact that the index of self-adjoint operator is 0?

Exercise 10.2. Any projective line intersects E in 3 points counting with multiplicity (for a point P near which E is complex-analytic: the multiplicity is 2 if the line is tangent to E at the given point; is 3 if the tangency is at the inflection point).

In what follows we assume that E is complex-analytic. (However, for the first steps one can proceed similarly in the general case if one restricts the attention to the complex-analytic part of E.)

Given 3 points $P_{1,2,3}$, say that $P_1 + P_2 + P_3 = 0$ (here + and 0 are formal symbols only!) iff the points are the points of intersection of a projective line with E (counted with multiplicities).

Exercise 10.3. Given a point $0 \in E$, E may be equipped with a group law + with the neutral element 0 which is compatible with the relation $P_1 + P_2 + P_3 = 0$ given above iff 0 is an inflection point of E. The group law + is uniquely defined.

Exercise 10.4. In the group structure as in the previous exercise, a point is a point of order 3 iff it is an inflection point.

In particular, an elliptic curve $Y^2 = P_3(X)$ has a group law for which P_∞ is the neutral element (if P_3 has simple roots). Consider this group law on this curve.

Exercise 10.5. The inversion on the group is the mapping $(X, Y) \to (X, -Y)$. The group has exactly 4 elements of order 2.

Exercise 10.6. The uniformization U from Exercise 7.5 indentifying E with $\mathbb{C}^2/L$ is a group isomorphism.

Exercise 10.7. There are exactly 9 inflection points on E.

Consider an arbitrary cubic curve E which contains P_∞ as an inflection point such that the tangent line is l_∞.

Exercise 10.8. There is a coordinate change $\widetilde{X} = AX + C$, $\widetilde{Y} = aX + bY + c$ and numbers $g_4, g_6 \in \mathbb{C}$ such that in the coordinates $\widetilde{X}$, $\widetilde{Y}$ the curve E can be written as $\widetilde{Y}^2 = 4\widetilde{X}^3 - g_4\widetilde{X} - g_6$. The coordinates $\widetilde{X}, \widetilde{Y}$ are defined uniquely up to a transformation $\widetilde{X}' = s\widetilde{X}^2$, $\widetilde{Y}' = s\widetilde{Y}^3$. Under such transformations $g_4' = s^4 g_4$, $g_6' = s^6 g_6$.

Label exer485

Exercise 10.9. The transformation $\widetilde{X} = \frac{AX+BY+C}{\alpha X+\beta Y+\gamma}$, $\widetilde{Y} = \frac{aX+bY+c}{\alpha X+\beta Y+\gamma}$ can be continuously extended to a bijection $\mathbb{CP}^2 \to \mathbb{CP}^2$ iff there is no relationship of the form $\varepsilon\widetilde{X} + \zeta\widetilde{Y} = \text{const}$. The set of solutions to an equation $R_k(X, Y) = 0$ goes to the set of solutions to an equation $\widetilde{R}_k\left(\widetilde{X}, \widetilde{Y}\right) = 0$; here R_k, $\widetilde{R}_k$ are polynomials of degree k without linear factors.

Given 4 distinct points $P_{1,2,3,4} \in \mathbb{CP}^1$ such that no triple is on the same projective line, and another such collection $Q_{1,2,3,4}$, one can find a unique transformation f as above such that $f(P_k) = Q_k$, $k = 1, \ldots, 4$.

Such transformation are called *projective transformations*. In fact, any complex analytic bijection $\mathbb{CP}^2 \to \mathbb{CP}^2$ has this form.

Exercise 10.10. Given a complex-analytic cubic curve C and an inflection point $P \in C$, there is a projective transformation f and a polynomial $P_3(X)$ of degree 3 with simple roots such that $f(P) = P_\infty$, and $f(C)$ is the projective completion of $\{Y^2 = P_3(X)\}$. In particular, if a cubic has an inflection point, it has exactly 9 of them.

Actually, any cubic has an inflection point; thus it is isomorphic to an elliptic curve, has genus 1, and has exactly 9 inflection points. To show this, one needs a little bit of enumerative geometry.

Exercise 10.11. Given a function $f(X, Y)$ of two variables, and a point P with coordinates (X, Y) such that $f(P) = 0$, denote by f_P restriction of P to the tangent line to the curve $C = \{f = 0\}$ on the plane. Suppose that C has no singularity at P; i.e., $df|_P \neq 0$. The point P is an inflection point for C iff f_P has a zero of the order ≥ 3 at P. Thus singular and inflection points coincide with common zeros of f and $f_{,y}^2 f_{,xx} - 2f_{,x}f_{,y}f_{,xy} + f_{,x}^2 f_{,yy}$; here lower indices denote the corresponding partial derivatives.

Exercise 10.12. Consider two projective curves $C, \widetilde{C} \subset \mathbb{CP}^2$ given by equations $P(X, Y) = 0$, $\widetilde{P}(X, Y) = 0$ correspondingly; here P and $\widetilde{P}$ are polynomials. Suppose that $C \cap C'$ is a finite set, and any point of this set is non-singular both on C and $\widetilde{C}$. Count points of $C \cap C'$ with "the usual" multiplicity (i.e., 1 if the tangent lines are different, 2 if there is a tangency with different curvatures, 3 if there is a tangency with the same curvature but different derivative of the curvature etc). Then the total count is $(\deg P) \cdot \left(\deg \widetilde{P}\right)$. *Hint: The equation holds if P and $\widetilde{P}$ are products of linear factors; moreover, the difference does not change if one deforms P or $\widetilde{P}$ such that all the conditions hold. Thus it is enough to show that any polynomial P can be deformed to a product of linear polynomials without introducing singular points on $\{P = 0\}$.*

Given a polynomial $P_3(X, Y)$, let $P_5 = f_{,y}^2 f_{,xx} - 2f_{,x}f_{,y}f_{,xy} + f_{,x}^2 f_{,yy}$; here $f = P_3$. We know that the intersection of curves $\{P_3 = 0\}$ and $\{P_5 = 0\}$ consists of 15 points counting the multiplicity. To show that there is an inflection point, it is enough to show that not all of the intersection points can be at the infinity l_∞ of $\mathbb{CP}^1$.

Since $P_3 = 0$ has at most 3 points on l_∞, and the total count with multiplicities is 15, it is enough to show that the multiplicity of a point on l_∞ is 4 or less.

Exercise 10.13. Consider a curve C passing through the point (0,0) which is not an inflection point on C. Suppose C is not tangent to $\{X = 0\}$ at (0,0), and that $g(X, Y) = 0$ is the equation of C near (0,0). Let $f(X, Y) = g(X, Y)/X^n$; let $h = f_{,y}^2 f_{,xx} - 2f_{,x}f_{,y}f_{,xy} + f_{,x}^2 f_{,yy}$; let $C' = \{f = 0\}$, $\widetilde{C}' = \{h = 0\}$. Then near (0,0) the curve C coincides with the closure of C'; denote by $\widetilde{C}$ the closure of $\widetilde{C}'$. Show that $(0,0) \in C \cap \widetilde{C}$ with multiplicity $(n-1)(n-2)$.

Exercise 10.14. Suppose that the projective completion of $\{P_3 = 0\}$ has 3 distinct points on l_∞ and none of these points is an inflection point. Show that the multiplicity of any of these points in intersection of $\{P_3 = 0\}$ and $\{P_5 = 0\}$ is 2.

Exercise 10.15. Show that any complex analytic cubic has an inflection point, is isomorphic to an elliptic curve, and has exactly 9 inflection points. *Hint: compare with Exercise* 10.9.

11. Elliptic quartics

Exercise 11.1. Consider the set E_0' of solution to $Y^2 = P_4(X)$ in $\mathbb{C}^2$; here P_4 is a polynomial of degree 4. Then for large enough R, the intersection of E_0' with the complement of a ball of radius R consists of two connected components. Denote these components by E_+ and E_-. Then the closure $\bar{E}_+$ of E_+ in $\mathbb{CP}^2$ is $E_+ \cup \{P_\infty\}$; $P_\infty = (0:1:0)$. $\bar{E}_+$ is a complex analytic manifold with boundary; the boundary is a subset of the sphere of radius R.

The tangent line to $\bar{E}_+$ at P_∞ coincides with l_∞; the order of tangency of l_∞ with $\bar{E}_+$ is 1 (so P_∞ is not an inflection point of $\bar{E}_+$).

Same statements are applicable to E_- too.

In other words, the closure E' of E_0' in $\mathbb{CP}^2$ is not complex analytic; $E' \smallsetminus E_0'$ consists of one point P_∞ through which *two branches* of the curve E' pass. These two branches are tangent to each other at P_∞.

Exercise 11.2. "Split" the point P_∞ on E'; in other words, construct a manifold E by gluing two points $P_\infty^\pm$ to E_0' so that the neighborhood of P_∞^+ is $P_\infty^+ \cup E_+ \simeq \bar{E}_+$, the neighborhood of P_∞^- is $P_\infty^- \cup E_- \simeq \bar{E}_-$. Then E is a complex analytic manifold; there is a natural mapping $E \to E'$ which is identical on $E_0' \subset E$, and sends $P_\infty^\pm$ to P_∞.

Then E is a torus.

This construction is usually called *normalization*; so E is the normalization of E'. One can make this construction less abstract: say that two curves C, C' on a plane are 2-equivalent at the point P, if $P \in C$, $P \in C'$, and C and C' have a tangency of 2nd order at P (i.e., C and C' have the same direction and the same curvature at P). A 2-*element* on a plane π is an equivalence class of curves at a point of the plane. The set $\operatorname{Jet}_2(\pi)$ of 2-elements forms a 4-dimensional manifold (two coordinates to specify P, direction and curvature). Any smooth curve C on the plane has a *lifting* C_2 to $\operatorname{Jet}_2(\pi)$; C_2 consists of 2-elements of C at points of C. The lifting is a smooth curve in $\operatorname{Jet}_2(\pi)$; the natural projection $\operatorname{Jet}_2(\pi) \to \pi_2$ sends C_2 to C.

In the construction above π may be a projective plane too; similarly, π may be a complex plane $\mathbb{C}^2$ (or $\mathbb{CP}^2$).

Exercise 11.3. Let $\widetilde{E}$ be the closure in $\operatorname{Jet}_2(\mathbb{CP}^2)$ of the lifting $(E_0')_2$. Then $\widetilde{E} \setminus (E_0')_2$ consists of two points $Q_\infty^\pm$; $\widetilde{E}$ is smooth in a neighborhood of these points. Thus $\widetilde{E}$ is isomorphic to E.

Exercise 11.4. Show that if E_0' is complex-analytic, E is diffeomorphic to a torus.

Exercise 11.5. Show that any meromorphic function on E can be written as $R(X,Y)$, and any meromorphic differential form on E can be written as $R(X,Y)\,dX$; here $R(X,Y)$ is a rational function of X and Y.

If E_0' is complex-analytic (in other words, P_4 has only simple zeros), then E is an elliptic curve. Taking a uniformization of this curve reduces any integral of the form $\int R\left(X, \sqrt{P_4(X)}\right) dX$ to the situation of Exercise 6.11.

References

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E-mail address: math@ilyaz.org