# SOLUTIONS FOR THE SAMPLE MIDTERM

**1**. Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y^2}$ if

$$f(x, y) = e^{xy} \tan(x).$$

**Solution.**

$$\frac{\partial f}{\partial x} = y e^{xy} \tan(x) + e^{xy} \sec^2(x)$$

$$\frac{\partial f}{\partial y} = x e^{xy} \tan(x)$$

$$\frac{\partial^2 f}{\partial x^2} = y^2 e^{xy} \tan(x) + 2y e^{xy} \sec^2(x) + 2e^{xy} \sec^2(x) \tan(x)$$

$$\frac{\partial^2 f}{\partial x \partial y} = (1 + xy) e^{xy} \tan(x) + x e^{xy} \sec^2(x)$$

$$\frac{\partial^2 f}{\partial y^2} = x^2 e^{xy} \tan(x)$$

**2**. Find all relative maxima and minima for the function

$$f(x, y) = \frac{1}{x} + \frac{1}{y} + xy.$$

**Solution.**

$$\frac{\partial f}{\partial x} = \frac{-1}{x^2} + y = 0$$

$$\frac{\partial f}{\partial y} = \frac{-1}{y^2} + x = 0$$

Solve the equations:

$$y = \frac{1}{x^2},\ x = \frac{1}{y^2}.$$

Get

$$y = \frac{1}{\left(\frac{1}{y^2}\right)^2} = y^4.$$

Since $x, y \neq 0$, one gets $y^3 = 1$, hence $x = y = 1$.

Now we apply the second derivative test

$$\frac{\partial^2 f}{\partial x^2} = \frac{2}{x^3},\ \frac{\partial^2 f}{\partial x \partial y} = 1,\ \frac{\partial^2 f}{\partial y^2} = \frac{2}{y^3}.$$

Hence $D(1, 1) = 4 - 1 = 3 > 0$, $\frac{\partial^2 f}{\partial x^2}(1, 1) = 2 > 0$, and (1,1) is a relative minimum.

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**3**. Find the point on the hyperbola $xy = 1$ that has the minimal distance to the point $(-1, 1)$.

**Solution.** Minimize the function of the square of the distance between a point $(x, y)$ on the hyperbola and the point $(-1, 1)$. This function is given by the formula

$$f(x, y) = (x + 1)^2 + (y - 1)^2 .$$

Use Lagrange's method with the constraint

$$g(x, y) = xy - 1.$$

Then

$$F(x, y, \lambda) = (x + 1)^2 + (y - 1)^2 + \lambda (xy - 1) .$$

Write two equations

$$\frac{\partial F}{\partial x} = 2(x + 1) + \lambda y = 0$$

$$\frac{\partial F}{\partial y} = 2(y - 1) + \lambda x = 0$$

Note that $xy = 1$, and therefore $x, y \neq 0$. Hence we have

$$\lambda = \frac{-2(x + 1)}{y} = \frac{-2(y - 1)}{x}.$$

Substitute $y = \frac{1}{x}$, $x = \frac{1}{y}$, obtain

$$-2(x + 1)x = -2(y - 1)y.$$

Simplify and get

$$x^2 + x = y^2 - y,$$

or

$$x + y = y^2 - x^2.$$

Note that $x$ and $y$ have the same sign, and therefore $x + y \neq 0$. Divide the equation on $x + y$ and get

$$y - x = 1.$$

Use $y = \frac{1}{x}$, multiply by $x$, get a quadratic equation

$$1 - x^2 = x.$$

It has two solutions

$$x_{1,2} = \frac{-1 \pm \sqrt{5}}{2}, \; y_{1,2} = \frac{1 \pm \sqrt{5}}{2}.$$

Both points

$$\left( \frac{-1 + \sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2} \right)$$

and

$$\left(\frac{-1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$$

give a solution because of the symmetry about the line $y=-x$.

**4**. Evaluate the integral

$$\iint_R xydydx$$

over the triangle $R$ with vertices (0,0),(2,0),(2,1).

**Solution.** Write the corresponding double integral

$$\int_0^2 \int_0^{x/2} xydydx.$$

Calculate the first integral

$$\int_0^{x/2} xydy = \frac{1}{2}xy^2\Big|_0^{x/2} = \frac{x^3}{8}.$$

Finish by calculating the second integral

$$\int_0^2 \frac{x^3}{8}dx = \frac{x^4}{32}\Big|_0^2 = \frac{16}{32} = \frac{1}{2}.$$

**5**. Evaluate the following integrals

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos x}dx;$$

$$\int_1^2 x\ln xdx;$$

$$\int_0^\infty \frac{2x}{(x^2+1)^3}dx.$$

**Solution.** In the first integral make the substitution $u = 1+\cos x$. Then

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos x}dx = -\int_2^1 \frac{1}{u}du = -\ln u\Big|_2^1 = \ln\ 2.$$

The second integral can be taken by parts

$$\int_1^2 x\ln xdx = \frac{x^2}{2}\ln x\Big|_1^2 - \int_1^2 \frac{x^2}{2}\frac{1}{x}dx = 2\ln 2 - \frac{x^2}{4}\Big|_1^2 = 2\ln 2 - 1 + \frac{1}{4} = 2\ln 2 - \frac{3}{4}.$$

To take the last integral make the substitution $u = x^2+1$. If $x\to\infty$, then $u\to\infty$. Therefore we have

$$\int_0^\infty \frac{2x}{(x^2+1)^3}dx = \int_1^\infty \frac{du}{u^3} = \lim_{b\to\infty}\int_1^b \frac{du}{u^3}$$

$$\int_1^b \frac{du}{u^3} = \frac{-1}{2u^2}\Big|_1^b = \frac{1}{2} - \frac{1}{2b^2}$$

$$\int_1^\infty \frac{du}{u^3} = \lim_{b\to\infty} \frac{1}{2} - \frac{1}{2b^2} = \frac{1}{2}$$

**6**. The population of a small town was 1000 in 1998, 2100 in 2000 and 4050 in 2004. Assuming that the population grows linearly predict the population in 2010.

**Solution.** Let $t$ denote the number of years passed from 1998. Assume that the population $f(t)$ is given by a linear function $Ax + B$. Use the formulas from Section 7.5 to find A and $B$.

$$\Sigma x = 0 + 2 + 6 = 8,\ \Sigma y = 1000 + 2100 + 4050 = 7150,$$

$$\Sigma x^2 = 0 + 4 + 36 = 40,\ \Sigma xy = 0 + 4200 + 24300 = 28500.$$

$$A = \frac{3\Sigma xy - \Sigma x \Sigma y}{3\Sigma x^2 - (\Sigma x)^2} = \frac{28300}{56} \approx 505.4$$

$$B = \frac{\Sigma y - A\Sigma x}{3} = 1036$$

Therefore

$$f(x) = 1036 + 505.4x,$$

and the population in 2100 should be

$$f(12) = 1036 + 505.4 \cdot 12 \approx 7100.$$