1. Using the method of Lagrange multipliers minimize the function $xy$ on the ellipse $x^2 + 4y^2 = 8$.

2. Evaluate the integral
   \[ \int \int_{R} (x^2 + y) \, dx \, dy, \]
   if $R$ is bounded by $x = 1$, $y = x$ and $y = 3x$.

3. Determine if the following integral is convergent and evaluate it if it is convergent
   \[ \int_{0}^{\infty} \frac{x^3}{x^4 + 2} \, dx. \]

4. Solve the initial value problem
   \[ y' = \frac{x}{y}, \quad y(1) = -1. \]

5. Let $y = f(x)$ be the solution of initial value problem
   \[ y' = e^x \left( y^2 - 1 \right), \quad y(0) = 0. \]
   Find $y'(0)$ and $y''(0)$.

6. Find the Taylor series at $x = 0$ of the function
   \[ f(x) = \frac{1}{(x^2 + 4)}. \]

7. Evaluate the sum of the following series
   \[ \sum_{k=1}^{\infty} \frac{2^{k+1}}{3^{k-1}} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{3^k k}. \]

8. Approximately 10% of middle school students need eye glasses. You chose three students randomly. Let $X$ be the number of students among these three that need eye glasses. Find the expected value of $X$.

9. Assuming that a life of a computer is an exponential random variable evaluate the probability that a computer will work longer than its expected life.

10. The probability of a rainy day in Berkeley in January is 60%. Find the probability that there will be no more than two rainy days before the sunny day.

---

\textit{Date:} May 16, 2005.

1
Solutions.

1. 
\[ F(x, y, \lambda) = xy + \lambda(x^2 + 4y^2 - 8) \]

\[ \frac{\partial F}{\partial x} = y + 2\lambda x = 0, \]
\[ \frac{\partial F}{\partial y} = x + 8\lambda y = 0, \]
\[ x^2 + 4y^2 = 8. \]

One can assume that \( x \neq 0 \) and \( y \neq 0 \). Indeed, if \( x = 0 \), then \( y = 0 \) and they cannot satisfy constraint equation. Then
\[ \lambda = -\frac{y}{2x} = -\frac{x}{8y} \]

Obtain
\[ 8y^2 = 2x^2, \quad x = \pm 2y. \]

Possible solutions \((2, 1), (2, -1), (-2, 1), (-2, -1)\). Minimum is at \((2, -1)\) and \((2, -1)\).

2. 
\[ \int_1^3 \int_x^{3x} (x^2 + y) \, dy \, dx \]

\[ \int_x^{3x} (x^2 + y) \, dy = x^2y + \frac{1}{2}y^2 |_x^{3x} = 3x^3 + \frac{9}{2}x^2 - x^3 - \frac{1}{2}x^2 = 2x^3 + 4x^2 \]

\[ \int_0^1 (2x^3 + 4x^2) \, dx = \frac{1}{2}x^4 + \frac{4}{3}x^3 |_0^{1} = \frac{11}{6} \]

3. Make a substitution \( u = x^4 + 2 \) and check that the integral is divergent.

4. 
\[ \int y \, dy = \int x \, dx \]

\[ y^2 = x^2 + C \]

If \( x = 1, \ y = -1 \), then \( C = 0 \). Therefore the solution is \( y = -x \).

5. 
\[ y'(0) = e^0 (0^2 - 1) = -1 \]
\[ y'' = \frac{d}{dx} \left( e^x (y^2 - 1) \right) = e^x (y^2 - 1) + e^x 2yy' \]
\[ y''(0) = e^0 (0^2 - 1) + e^0 ( -1) = -1 \]

6. Put \( u = \frac{x^2}{4} f(x) = \frac{1}{x^2+4} = \frac{1}{4(x/2)^2+1} = \frac{1}{4(1+u^2)} \)

Since
\[ \frac{1}{1+u} = 1 - u + u^2 - u^3 + \ldots \]
we have
\[ f(x) = \frac{1}{4} - \frac{x^2}{16} + \frac{x^4}{64} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{4^{k+1}} + \cdots \]

7. \[ \sum_{k=1}^{\infty} \frac{2^{k+1}}{5^{k-1}} = \frac{4}{1 - \frac{2}{5}} = \frac{20}{3} \]

For the second series use
\[ \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \]

If \( x = -\frac{1}{3} \), then
\[ \ln \left( 1 - \frac{1}{3} \right) = -\sum_{k=1}^{\infty} \frac{1}{3^k} \]

Hence
\[ \sum_{k=1}^{\infty} \frac{1}{3^k} = -\ln \left( 1 - \frac{1}{3} \right) = -\ln \frac{2}{3} = \ln 3 - \ln 2 \]

8. The random variable \( X \) has values 0,1,2,3 and the probabilities are
\[ p_0 = \left( \frac{9}{10} \right)^3, \quad p_1 = 3 \left( \frac{1}{10} \right) \left( \frac{9}{10} \right)^2, \quad p_2 = 3 \left( \frac{1}{10} \right)^2 \left( \frac{9}{10} \right), \quad p_3 = \left( \frac{1}{10} \right)^3 \]

Then \( E(X) = p_1 + 2p_2 + 3p_3 = 0.3 \)

This can be explained in the simpler way: 10% of 3 is 0.3

9. If \( N \) be the expected life, then the probability density function is given by the formula
\[ f(x) = \frac{1}{N} e^{-\frac{x}{N}} \]

and
\[ \Pr(X > N) = 1 - \Pr(X \leq N) = 1 - \int_0^N \frac{1}{N} e^{-\frac{x}{N}} dx = e^{-1} \approx 0.368 \]

10. Here we have a geometric random variable. If \( p_n \) is the number of rainy days before the sunny day, then
\[ p_n = (0.6)^n (0.4) \]

Hence the probability that there will be no more than two rainy days is
\[ p_0 + p_1 + p_2 = 0.4 + (0.6) (0.4) + (0.6)^2 (0.4) = 0.4 \frac{1 - (0.6)^3}{1 - 0.6} = 0.784. \]

Another solution: the probability of three rainy days in a row is \((0.6)^3\). The probability that there is no more than 2 rainy days is \(1 - (0.6)^3\).