PRACTICE FINAL 3
MATH 16b

1. Using the method of Lagrange multipliers minimize the function xy on the ellipse $x^2 + 4y^2 = 8$.

2. Evaluate the integral

$$\iint_R \left(x^2 + y\right) dxdy,$$

if R is bounded by $x = 1$, $y = x$ and $y = 3x$.

3. Determine if the following integral is convergent and evaluate it if it is convergent

$$\int_0^\infty \frac{x^3}{x^4 + 2} dx.$$

4. Solve the initial value problem

$$y' = \frac{x}{y},\ y(1) = -1.$$

5. Let $y = f(x)$ be the solution of initial value problem

$$y' = e^x \left(y^2 - 1\right),\ y(0) = 0.$$

Find $y'(0)$ and $y''(0)$.

6. Find the Taylor series at $x = 0$ of the function

$$f(x) = \frac{1}{(x^2 + 4)}.$$

7. Evaluate the sum of the following series

$$\sum_{k=1}^\infty \frac{2^{k+1}}{5^{k-1}}$$

$$\sum_{k=1}^\infty \frac{1}{3^k k}$$

8. Approximately 10% of middle school students need eye glasses. You chose three students randomly. Let X be the number of students among these three that need eye glasses. Find the expected value of X.

9. Assuming that a life of a computer is an exponential random variable evaluate the probability that a computer will work longer than its expected life.

10. The probability of a rainy day in Berkeley in January is 60%. Find the probability that there will be no more than two rainy days before the sunny day.

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Solutions.

1.

$$F(x,y,\lambda) = xy + \lambda\left(x^2+4y^2-8\right)$$
$$\frac{\partial F}{\partial x} = y + 2\lambda x = 0,$$
$$\frac{\partial F}{\partial y} = x + 8\lambda y = 0,$$
$$x^2+4y^2 = 8.$$

One can assume that $x \neq 0$ and $y \neq 0$. Indeed, if $x = 0$, then $y = 0$ and they can not satisfy constraint equation. Then

$$\lambda = -\frac{y}{2x} = -\frac{x}{8y}$$

Obtain

$$8y^2 = 2x^2,\ x = \pm 2y.$$

Possible solutions $(2,1), (2,-1), (-2,1), (-2,-1)$. Minimum is at $(2,-1)$ and $(2,-1)$.

2.

$$\int_0^1\int_x^{3x}\left(x^2+y\right)dydx$$
$$\int_x^{3x}\left(x^2+y\right)dy = x^2y+\frac{1}{2}y^2|_x^{3x} = 3x^3+\frac{9}{2}x^2-x^3-\frac{1}{2}x^2 = 2x^3+4x^2$$
$$\int_0^1\left(2x^3+4x^2\right)dx = \frac{1}{2}x^4+\frac{4}{3}x^3|_0^1 = \frac{11}{6}$$

3. Make a substitution $u = x^4+2$ and check that the integral is divergent.

4.

$$\int ydy = \int xdx$$
$$y^2 = x^2 + C$$

If $x = 1$, $y = -1$, then $C = 0$. Therefore the solution is $y = -x$.

5.

$$y'(0) = e^0\left(0^2-1\right) = -1$$
$$y'' = \frac{d}{dx}\left(e^x\left(y^2-1\right)\right) = e^x\left(y^2-1\right) + e^x 2yy'$$
$$y''(0) = e^0\left(0^2-1\right) + e^0 0\,(-1) = -1$$

6. Put $u = \frac{x^2}{4} f(x) = \frac{1}{x^2+4} = \frac{1}{4}\frac{1}{(x/2)^2+1} = \frac{1}{4}\frac{1}{1+u}$

Since

$$\frac{1}{1+u} = 1 - u + u^2 - u^3 + \dots$$

we have

$$f(x) = \frac{1}{4} - \frac{x^2}{16} + \frac{x^4}{64} - \dots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{4^{k+1}} + \dots$$

7.

$$\sum_{k=1}^{\infty} \frac{2^{k+1}}{5^{k-1}} = \frac{4}{1-\frac{2}{5}} = \frac{20}{3}$$

For the second series use

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

If $x = -\frac{1}{3}$, then

$$\ln\left(1 - \frac{1}{3}\right) = -\sum_{k=1}^{\infty} \frac{1}{3^k k}$$

Hence

$$\sum_{k=1}^{\infty} \frac{1}{3^k k} = -\ln\left(1 - \frac{1}{3}\right) = -\ln\frac{2}{3} = \ln 3 - \ln 2$$

8. The random variable X has values 0,1,2,3 and the probabilities are

$$p_0 = \left(\frac{9}{10}\right)^3, \; p_1 = 3\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^2, \; p_2 = 3\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right), \; p_3 = \left(\frac{1}{10}\right)^3$$

Then $E(X) = p_1 + 2p_2 + 3p_3 = 0.3$

This can be explained in the simpler way: 10% of 3 is 0.3

9. If N be the expected life, then the probability density function is given by the formula

$$f(x) = \frac{1}{N} e^{\frac{-x}{N}}$$

and

$$\Pr(X > N) = 1 - \Pr(X \le N) = 1 - \int_0^N \frac{1}{N} e^{\frac{-x}{N}} dx = e^{-1} \approx 0.368$$

10. Here we have a geometric random variable. If p_n is the number of rainy days before the sunny day, then

$$p_n = (0.6)^n (0.4).$$

Hence the probability that there will be no more than two rainy days is

$$p_0 + p_1 + p_2 = 0.4 + (0.6)(0.4) + (0.6)^2(0.4) = 0.4\frac{1-(0.6)^3}{1-0.6} = 0.784.$$

Another solution: the probability of three rainy days in a row is $(0.6)^3$. The probability that there is no more than 2 rainy days is $1 - (0.6^3)$.