PRACTICE FINAL 2
MATH $16b$

1. Find possible relative maxima and minima of the function
$$f(x, y) = x^3 + 3x^2y + y^2 + x^2.$$
In each case apply the second derivative test if possible.

2. Evaluate the integral
$$\int_0^\infty \frac{x^2}{\left(x^3+1\right)^{100}} dx.$$

3. Describe the shape of a cylindrical can of a given volume that minimizes the surface area.

4. Solve the initial value problem
$$y' - 3xy = x,\ y(0) = 1.$$

5. Frog population in a pond grows with monthly intrinsic rate 0.5. The carrying capacity of the pond is 300. Write down differential equation describing the growth of the population and find the rate of growth at the moment when there are 100 frogs in the pond.

6. Find the function $f(x)$ such that the Taylor series of $f(x)$ at $x = 0$ is
$$\sum_{k=1}^{\infty} kx^{2k}.$$

7. A patient gets $5mg$ injection each day. The body eliminates 30% after 24 hours. Evaluate the amount of drug after extended treatment at the moment just after injection.

8. Let X be a random variable which takes values on the interval $[-1, 1]$. Find $E(X)$ and $\operatorname{Var}(X)$ if the density function
$$f(x) = \frac{3}{4}\left(1 - x^2\right).$$

9. Scores on a school's entrance exam are normally distributed with $\mu = 500$ and $\sigma = 100$. If the school wishes to admit only the students in the top 40%, what should be the cutoff grade?

10. The number of typos on a page of a certain document has a Poisson distribution. There is an average of two typos per page.
(a) Find the probability that a randomly selected page is typo free.
(b) Find the probability that the page has at least 1 typo.

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Solutions.

1. We have to solve the equations

$$\frac{\partial f}{\partial x} = 3x^2 + 6xy + 2x = x\,(3x + 6y + 2) = 0,$$

$$\frac{\partial f}{\partial y} = 3x^2 + 2y = 0.$$

From the first equation we get either $x = 0$ or $3x + 6y + 2 = 0$.

If $x = 0$, then $y = 0$ from the second equation. Apply the second derivative test

$$\frac{\partial^2 f}{\partial x^2} = 6x + 6y + 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6x$$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

$$D\,(0,0) = 4,\ \frac{\partial^2 f}{\partial y^2}\,(0,0) = 2 > 0.$$

Hence (0,0) is a local minimum.

Let $x \neq 0$. Then

$$3x + 6y + 2 = 0,\ 3x^2 + 2y = 0.$$

The second equation gives $y = \frac{-3x^2}{2}$. Substitute into the first equation

$$3x - 9x^2 + 2 = 0.$$

Solve this equation and obtain the following solutions

$$x = \frac{2}{3},\ y = -\frac{2}{3}$$

$$x = -\frac{1}{3},\ y = -\frac{1}{6}$$

Apply the second derivative test

$$D\left(\frac{2}{3}, -\frac{2}{3}\right) = 0,$$

$$D\left(-\frac{1}{3}, -\frac{1}{6}\right) = -6 < 0.$$

In the second case we see that $\left(-\frac{1}{3}, -\frac{1}{6}\right)$ is neither maximum nor minimum. In the first case the test does not work.

2. Use the substitution $u = x^3 + 1$, $du = 3x^2 dx$.

$$\int_0^b \frac{x^2}{(x^3+1)^{100}} dx = \int_1^{b^3+1} \frac{1}{3u^{100}} du == -\frac{1}{297u^{99}}\Big|_1^{b^3+1} = \frac{1}{297} - \frac{1}{297\,(b^3+1)^{99}}$$

$$\lim_{b\to\infty}\frac{1}{297\left(b^3+1\right)^{99}}=0.$$

Then

$$\int_0^\infty \frac{x^2}{\left(x^3+1\right)^{100}}dx=\frac{1}{297}.$$

3. A cylindrical can has two parameters, the radius r and the height h. The volume is fixed, i.e.

$$\pi r^2 h=V$$

and we have to minimize the surface area

$$A=2\pi r^2+2\pi\,\mathrm{rh}\,.$$

One can use Lagrangian multipliers. Divide both equations by π to simplify algebra.

$$F\left(r,h,\lambda\right)=2r^2+2rh+\lambda\left(r^2h-\frac{V}{\pi}\right).$$

We have to solve

$$\frac{\partial F}{\partial r}=4r+2h+2\lambda rh=0$$

$$\frac{\partial F}{\partial h}=2r+\lambda r^2=0.$$

The second equation gives $\lambda=-\frac{2}{r}$. Substitute into the first equation and get

$$4r+2h-4h=0$$

or $h=2r$. Hence the height must be equal to the diameter.

4. First find the integrating factor $e^{\frac{-3x^2}{2}}$. Multiply the equation on the integrating factor

$$y'e^{\frac{-3x^2}{2}}-3xe^{\frac{-3x^2}{2}}y=xe^{\frac{-3x^2}{2}}$$

$$\left(ye^{\frac{-3x^2}{2}}\right)'=xe^{\frac{-3x^2}{2}}$$

$$ye^{\frac{-3x^2}{2}}=\int xe^{\frac{-3x^2}{2}}dx=-\frac{1}{3}e^{\frac{-3x^2}{2}}+C$$

$$y=-\frac{1}{3}+Ce^{\frac{3x^2}{2}}$$

To find C use that $y\left(0\right)=1$. Then $C=\frac{4}{3}$ and

$$y=-\frac{1}{3}+\frac{4}{3}e^{\frac{3x^2}{2}}.$$

5. Differential equation on the size of population is

$$y'=0.5\frac{y\left(300-y\right)}{300}.$$

When $y = 100$, the rate of growth is

$$y' = 0.5\frac{100\left(300-100\right)}{300} \approx 33.3.$$

The rate 33.3 frogs per month.

6. Let $y = x^2$. Then

$$f\left(y\right) = \sum_{k=1}^{\infty} k y^k$$

$$\frac{f\left(y\right)}{y} = \sum_{k=1}^{\infty} k y^{k-1} = 1 + 2y + 3y^2 + 4y^3 + \dots$$

Now note that

$$1 + 2y + 3y^2 + 4y^3 + \cdots = \left(1 + y + y^2 + y^3 + \dots\right)' = \left(\frac{1}{1-y}\right)' = \frac{1}{\left(1-y\right)^2}.$$

Substitute back $y = x^2$ and obtain the answer

$$f\left(x\right) = \frac{1}{\left(1-x^2\right)^2}.$$

7. The answer is given by the sum of the following geometric series

$$5 + 0.7 \times 5 + 0.7^2 \times 5 + 0.7^3 \times 5 + \cdots = \frac{5}{1-0.7} = \frac{50}{3} \approx 16.67mg$$

8.

$$E\left(X\right) = \int_{-1}^{1} \frac{3}{4} x\left(1-x^2\right) dx = 0$$

$$\mathrm{Var}\left(X\right) = \int_{-1}^{1} \frac{3}{4} x^2\left(1-x^2\right) dx = \frac{3}{4}\int_{-1}^{1} x^2 - x^4 dx = \frac{x^3}{3} - \frac{x^5}{5}\Big|_{-1}^{1} = \frac{4}{15}$$

9. The density function is $f\left(x\right) = \frac{1}{100\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{(x-500)}{100}\right)^2}$
We have to find t such that

$$\Pr\left(X > t\right) = 0.4.$$

On the other hand

$$\Pr\left(X > t\right) = \Pr\left(X \geq 500\right) - \Pr\left(500 \leq X \leq t\right)$$

But $\Pr\left(X \geq 500\right) = 0.5$. Hence

$$\Pr\left(X > t\right) = 0.5 - \Pr\left(500 \leq X \leq t\right).$$

Therefore

$$\Pr\left(500 \leq X \leq t\right) = 0.1.$$

We have the formula

$$\Pr(500 \le X \le t) = \int_{500}^{t} \frac{1}{100\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{(x-500)}{100}\right)^2} dx.$$

Make the substitution $z = \frac{x-500}{100}$, let $s = \frac{t-500}{100}$.

$$\Pr(500 \le X \le t) = \int_{0}^{s} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = A(s).$$

From the table for $A(z)$ find that $A(s) = 0.1$ if $s \approx 0.26$. Then

$$t = 100s + 500 = 526.$$

10. The Poisson distribution in this case has $\lambda = 2$. Therefore

$$p_n = e^{-2}\frac{2^n}{n!}.$$

Then $p_0 = e^{-2} \approx 13.5\%$ is the probability that a random page is typo free and $1 - p_0 \approx 86.5\%$ is the probability that a page has at least one typo.