PRACTICE FINAL 1
MATH 16b

1. Let
$$f(x,y,z) = x^3 - y\tan z.$$
Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$.

2. Evaluate the integral
$$\iint_R (x+y)\,dxdy$$
for R bounded by the line $y = x$ and the parabola $y = x^2$.

3. Use partial derivatives to obtain the line equation which is the best to fit the data points $(-1,-1), (0,1), (1,2)$.

4. Solve the initial value problem
$$y' = y^2 \sin x,\ y(0) = 8.$$

5. A bank account has \$50,000 earning 5% interest compounded continuously. An owner of the account withdraws money continuously at the annual rate of \$4,000. How long will it take for the balance to drop to zero?

6. Approximate
$$\int_0^1 \cos\left(x^2\right) dx.$$

7. Check for convergence
$$(a) \sum_{k=1}^{\infty} \frac{1}{k^3+3k+5},$$
$$(b) \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+1}}.$$

8. Find $f^{(7)}(0)$ for
$$f(x) = \frac{\cos(x^2) - 1}{x}.$$

9. Suppose that the amount of time required to serve a customer in a bank has an exponential density function with mean 10 minutes. Find the probability that a customer will require more than 10 minutes.

10. A student can take a certain exam three times. If he fails all three times, he should leave the school. The probability that he passes exam is 80%. Find the probability that he will stay at this school.

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Solutions.

1.

$$\frac{\partial f}{\partial x} = 3x^2,\ \frac{\partial f}{\partial y} = -\tan z,\ \frac{\partial f}{\partial z} = \frac{y}{\cos^2 z}.$$

2. Rewrite the integral as

$$\int_0^1 \int_{x^2}^x (x+y)\, dydx.$$

Then

$$\int_{x^2}^x (x+y)\, dy = xy + \frac{y^2}{2}\Big|_{x^2}^x = x^2 - x^3 + \frac{x^2 - x^4}{2} = \frac{3x^2}{2} - x^3 - \frac{x^4}{2}$$

$$\int_0^1 \left(\frac{3x^2}{2} - x^3 - \frac{x^4}{2} \right) dx = \frac{x^3}{2} - \frac{x^4}{4} - \frac{x^5}{10}\Big|_0^1 = \frac{3}{20}$$

3. Write the equation $y = Ax + B$. Then we have to minimize

$$E = (B - A + 1)^2 + (B - 1)^2 + (B + A - 2)^2 = 3B^2 + 2A^2 - 4B - 6A + 6.$$

Write the equation for partial derivatives

$$\frac{\partial E}{\partial A} = 4A - 6 = 0$$

$$\frac{\partial E}{\partial B} = 6B - 4 = 0$$

Solve and obtain $A = \frac{3}{2}$, $B = \frac{2}{3}$. The equation is

$$y = \frac{3}{2}x + \frac{2}{3}.$$

4. First find a general solution

$$\int \frac{dy}{y^2} = \int \sin x dx,\ y = 0$$

$$-\frac{1}{y} = -\cos x + C$$

$$y = \frac{1}{\cos x - C}$$

To find C put $x = 0$, $y = 8$ and get the equation

$$8 = \frac{1}{1 - C}$$

Solve $C = \frac{7}{8}$.

The answer

$$y = \frac{1}{\cos x - 7/8}.$$

5. Differential equation on the account balance $y(t)$ is

$$y' = 0.05y - 4,000,\ y(0) = 50,000$$

First solve the equation. Integration factor is $e^{-0.05t}$.

$$\left(e^{-0.05t}y\right)' = -4,000e^{-0.05t}$$

$$e^{-0.05t}y = 80,000e^{-0.05t} + C$$

$$y = 80,000 + Ce^{0.05t}$$

Use initial data $y(0) = 50,000$ to find $C = -30,000$. Therefore the account balance after t years is given by the formula

$$y = 80,000 - 30,000e^{0.05t}.$$

To answer the problem question solve for t the equation

$$y = 80,000 - 30,000e^{0.05t}$$

$$e^{0.05t} = \frac{8}{3}$$

$$0.05t = \ln\frac{8}{3}$$

$$t = 20\ln\frac{8}{3} \approx 19.6.$$

6. Write the Taylor series for $\cos(x^2)$ at $x = 0$

$$\cos\left(x^2\right) = 1 - \frac{x^4}{2} + \frac{x^8}{24} - \dots$$

Then write the series for an antiderivative

$$\int \cos\left(x^2\right) dx = x - \frac{x^5}{10} + \frac{x^9}{216} - \dots$$

$$\int_0^1 \cos\left(x^2\right) dx \approx 1 - \frac{1}{10} + \frac{1}{216} \approx 0.905$$

7. In both cases use the comparison test. In the first case

$$\frac{1}{k^3 + 3k + 5} \leq \frac{1}{k^3}.$$

The series

$$\sum_{k=1}^{\infty} \frac{1}{k^3}$$

is convergent. Hence

$$\sum_{k=1}^{\infty} \frac{1}{k^3 + 3k + 5}$$

is convergent.

In the second case note that

$$\sqrt{k^2+1} \leq \sqrt{k^2+2k+1} = k+1.$$

Therefore

$$\frac{1}{\sqrt{k^2+1}} \geq \frac{1}{k+1}.$$

Since

$$\sum_{k=1}^{\infty} \frac{1}{k+1}$$

is divergent then

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+1}}$$

is divergent.

8.Use the Taylor series

$$f(x) = -\frac{x^3}{2} + \frac{x^7}{24} - \dots$$

If a_7 is the coefficient then $a_7 = \frac{f^{(7)}(0)}{7!}$. In our case $a_7 = \frac{1}{24}$. Hence

$$f^{(7)}(0) = \frac{7!}{24} = 210.$$

9. The density function is $f(x) = \frac{1}{10} e^{-\frac{x}{10}}$.

$$\Pr(X > 10) = 1 - \Pr(X \leq 10)$$

$$\Pr(X \leq 10) = \int_0^{10} \frac{1}{10} e^{-\frac{x}{10}} dx = -e^{-\frac{x}{10}}\Big|_0^{10} = 1 - e^{-1}.$$

$$\Pr(X > 10) = e^{-1} \approx 0.37.$$

10. The probability that he fails once is 0.2. The probability that he fails three times is $(0.2)^3 = 0.008$. Hence he will stay with probability $1 - 0.008 = 0.992 = 99.2\%$.