PRACTICE MIDTERM
MATH 130

1. Show that on a Hilbert plane every segment has a midpoint.

2. Let Φ be a rigid motion of a Hilbert plane such that Φ^2 is the identical map but Φ is not. Prove that Φ is a reflection or a central symmetry.

3. Let F be the set of all real numbers which can be written in the form $a + b\sqrt{3}$ for some rational a and b.

(a) Prove that F is an ordered field;

(b) Is Π_F a Hilbert plane?

4. Let Φ be a bijective transformation of an Euclidean plane which maps a line to a line and preserves betweenness. Prove that Φ maps the midpoint of a segment to the midpoint of a segment. (Attention: Φ may not preserve congruence.)

5. Consider the real Cartesian plane with origin removed. Define lines, betweenness and congruence on this set in the same way as for usual Cartesian plane. Which of the Hilbert axioms do not hold?

Solutions.

1. Consider a segment AB. It is proven in Hartshorne that there exists an isosceles triangle ABC with base AB. Let C' be the image of C under reflection in AB. (Reflection exists by (ERM).) The segment CC' meets the line AB at the point D. Since CC' is preserved by the reflection it is perpendicular to AB and $CD \simeq C'D$. Note that D is between A and B. Indeed, assume the opposite, say $A * B * D$, then $CB < CA$, and ACB is not isosceles. Now the triangles ACD and $BC'D$ are congruent by (RASS). Hence $AD \simeq DB$.

2. Pick up a point A such that $\Phi(A) \neq A$. Let $\Phi(A) = B$. Then $\Phi(B) = A$ because Φ^2 is identity. Let M be the midpoint of AB and l be the perpendicular bisector to AB. Then $\Phi(M) = M$ and $\Phi(l) = l$. Every point X on l either transforms to itself or to the point X' on the opposite side of M. In the first case every point X on l is fixed, so Φ is the reflection in l. In the second case Φ is the central symmetry with center M.

3. (a) we have to check that F is closed under addition, multiplication and inverses. Than all axioms of a field hold for F as they hold for $\mathbb{R}$.

$$(a + b\sqrt{3}) + (c + d\sqrt{3}) = ((a + c) + (b + d)\sqrt{3},$$
$$-(a + b\sqrt{3}) = -a - b\sqrt{3},$$
$$(a + b\sqrt{3})(c + d\sqrt{3}) = (ac + 3bd) + (ad + bc)\sqrt{3},$$
$$(a + b\sqrt{3})^{-1} = \frac{a}{a^2 - 3b^2} - \frac{b}{a^2 - 3b^2}\sqrt{3}.$$

(b). No, because F is not Pythagorean. For example $\sqrt{2}$ is not in F.

4. Consider a segment AB and let $A'B'$ be the image of AB under Φ. Consider a parallelogram $ACBD$ with diagonal AB. Since Φ is a bijection, Φ transforms parallel lines to parallel lines. Hence the image $A'C'B'D'$ is again a parallelogram. So the midpoint M of AB is the intersection of the diagonals of $ACBD$. Then $M' = \Phi(M)$ is the point of intersection of diagonals of $A'C'B'D'$. Hence M' is the midpoint of $A'B'$.

5. Only (B4) and (C1) do not hold. For all others the proof goes exactly as for $\Pi_{\mathbb{R}}$. To see that (B4) does not always hold take $A = (1, 0)$, $B = (1, 2)$, $C = (-1, 0)$, $D = (1, 1)$. The line $y = x$ passes through D but does not meet AC and BC since $(0, 0)$ is not on our "plane". For (C1) counter example take $A = (1, 0)$, $B = (2, 0)$, $C = A$ and a ray to be opposite of the ray AB.