PROBLEM SET # 5
MATH 114

Due February 23.

1. Determine which of the following polynomials are irreducible over \( \mathbb{Q} \):
   \[ x^4 + x^2 + 1, \ x^{12} + 99, \ x^3 + 2x + 1. \]

2. List all irreducible polynomials of degree 4 in \( \mathbb{Z}_2 \)[\(x\)].

3. Let \( F \) be a field, \( f(x), g(x) \in F[x] \) and \( f(\alpha) = g(\alpha) \) for any \( \alpha \in F \). Prove that if \( F \) is infinite then \( f(x) = g(x) \). Show that if \( F \) is finite, then the statement is wrong.

4. Let \( p \) be a prime number. Prove that \( f(x) = x^{p-1} + x^{p-2} + \cdots + 1 \) is irreducible over \( \mathbb{Q} \). Hint: first check that \( f(x) \) is irreducible if and only if \( f(x + 1) \) is irreducible. Use \( f(x) = \frac{x^p - 1}{x - 1} \). Prove that \( f(x + 1) \) is irreducible by Eisenstein criterion.

5. Find \( \left( \mathbb{Q}(\sqrt[3]{7}, \sqrt[17]{22}) / \mathbb{Q} \right) \).

6. Check that \( \mathbb{Z}_{11}(\sqrt{2}) \) and \( \mathbb{Z}_{11}(\sqrt{7}) \) are isomorphic.

7. Find the minimal polynomial for \( \sqrt{7} + \sqrt{3} \) over \( \mathbb{Q} \).