Due February 16.
Read chapter 1 in Artin’s book. We assume that all fields are commutative in homework problems.

1. Let $F$ be a field, and $F[i]$ denote the set of all expressions $a + bi$, with $a, b \in F$. Define addition and multiplication in $F[i]$ by

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$
$$(a + bi)(c + di) = ac - bd + (ad + bc)i.$$ Determine if $F[i]$ is a field for $F = \mathbb{Q}, \mathbb{R}, \mathbb{Z}_3, \mathbb{Z}_5$.

2. Assume that $\text{char } F = p$. Prove that $(a + b)^p = a^p + b^p$. Hint: use binomial formula.

3. Prove the little Fermat’s theorem

$$a^p \equiv a \mod p$$
for any prime $p$ and integer $a$. Hint: use the previous problem.

4. Let $V$ be a vector space of dimension $n$ and $A : V \rightarrow V$ be a linear map such that $A^N = 0$ for some integer $N > 0$. Prove that $A^n = 0$. Hint: check that $\text{Im } A^k$ is a proper subspace in $\text{Im } A^{k-1}$.

5. Find a formula for a general term of the Fibonacci sequence

1, 1, 2, 3, 5, 8, 13, ...

Hint: write the Fibonacci sequence as a linear combination of

$$1, \alpha, \alpha^2, \alpha^3, \ldots \text{ and } 1, \beta, \beta^2, \beta^3, \ldots,$$

where

$$\alpha = \frac{1 + \sqrt{5}}{2}, \beta = \frac{1 - \sqrt{5}}{2}.$$ 


(a) Prove that the number of one dimensional subspaces in $F^n$ equals $\frac{p^n - 1}{p - 1}$;

(b) (Extra credit) Find the number of 2-dimensional subspaces in $F^n$.

Date: February 9, 2006.