Due February 2.

1. An automorphism of a group $G$ is an isomorphism from $G$ to itself. Denote by $\text{Aut} \, G$ the set of all automorphisms of $G$.
   (a) Prove that $\text{Aut} \, G$ is a group with respect to the operation of composition.
   (b) Let $G$ be a finite cyclic group. Describe $\text{Aut} \, G$.
   (c) Give an example of an abelian $G$ such that $\text{Aut} \, G$ is not abelian.

2. Use the same notations as in Problem 1. Let $\pi_g$ be the map of $G$ to itself defined by $\pi_g(x) = gxg^{-1}$, here $g \in G$.
   (a) Show that $\pi_g \in \text{Aut} \, G$.
   (b) Let $\text{Inn} \, G = \{ \pi_g \mid g \in G \}$. Show that $\text{Inn} \, G$ is a normal subgroup in $\text{Aut} \, G$.

3. Show that a group of order $p^2$ is abelian.

4. One makes necklaces from black and white beads. Let $p$ be a prime number. Two necklaces are the same if one can be obtained from another by a rotation or a flip over. How many different necklaces of $p$ beads one can make?

5. Assume that $N$ is a normal subgroup of a group $G$. Prove that if $N$ and $G/N$ are solvable, then $G$ is solvable.

6. For any permutation $s$ denote by $F(s)$ the number of fixed points of $s$ ($k$ is a fixed point if $s(k) = k$). Let $N$ be a normal subgroup of $A_n$. Choose a non-identical permutation $s \in N$ with maximal possible $F(s)$.
   (a) Prove that any of disjoint cycles of $s$ has length not greater than 3. (Hint: if $s \in N$, then $gsg^{-1} \in N$ for any even permutation $g$).
   (b) Prove that the number of disjoint cycles in $s$ is not greater than 2.
   (c) Assume that $n \geq 5$. Prove that $s$ is a 3-cycle.
   (d) Use (c) to show that $A_n$ is simple for $n \geq 5$, i.e. $A_n$ does not have proper non-trivial normal subgroups. (Hint: $A_n$ is generated by 3-cycles, as it was proven in class).

Date: January 25, 2006.