Due April 6.

1. Let $n = p$, or $2p$ where $p$ is a prime number. Prove that the Galois group of the polynomial $x^n - 1$ over any field $F$ is cyclic.

2. Show that the Galois group of $x^{15} - 1$ over $\mathbb{Q}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$.

By $\mathbb{F}_q$ we denote the finite field of $q$ elements.

3. Find the Galois groups of $x^6 - 1$ over $\mathbb{F}_5$, $\mathbb{F}_{25}$ and $\mathbb{F}_{125}$.

4. Let $F \subset E$ be an extension of finite fields. Prove that
$$|E| = |F|^{[E/F]}.$$

5. Let $f(x) \in \mathbb{Z}_p[x]$ be an irreducible polynomial of degree 3. Prove that $f(x)$ is irreducible over $\mathbb{F}_{p^2}$.

6. Let $q = p^k$ for some prime $p$, $n$ be a number relatively prime to $p$, $m$ be the minimal positive integer such that
$$q^m \equiv 1 \pmod{n}.$$

Show that the Galois group of $x^n - 1$ over $\mathbb{F}_q$ is isomorphic to $\mathbb{Z}_m$.