REVIEW EXERCISES
MATH 114

1. Let $G$ be a transitive subgroup of $S_n$.
   \(a\) Prove that if $n$ is prime, then $G$ contains an $n$-cycle.
   \(b\) Show that \(a\) is not true if $n$ is not prime.

2. Let $F$ be a field such that the multiplicative group $F^*$ is cyclic. Prove that $F$ is finite.

3. Let $G$ be a transitive subgroup of $S_6$ which contains a 5-cycle. Prove that $G$ is not solvable.

4. Let $F$ be a field and $\text{char } F \neq 2$, $\alpha, \beta \in F$. Prove that $F(\sqrt{\alpha}) = F(\sqrt{\beta})$ if and only if $\alpha \beta$ is a square in $F$.

5. Find the minimal polynomial for $1 + 3\sqrt[3]{2} + 3\sqrt[3]{4}$ over $\mathbb{Q}$.

6. Prove that any algebraically closed field is infinite.

7. Is $x^3 + x + 1$ irreducible over $\mathbb{F}_{256}$?

8. Which of the following extensions are normal
   
   $\mathbb{Q} \subset \mathbb{Q}\left(\sqrt[3]{2}, \sqrt{3}\right)$
   
   $\mathbb{Q} \subset \mathbb{Q}\left(\sqrt[3]{2}, \sqrt{-3}\right)$?

9. Determine if
   
   $\mathbb{Q}\left(\sqrt[4]{1 - \sqrt{2}}\right) = \mathbb{Q}\left(\sqrt{-1}, \sqrt{2}\right)$.

10. Let $\mathbb{Q} \subset F$ be a finite normal extension such that for any two subfields $E$ and $K$ of $F$ either $K \subset E$ or $E \subset K$. Then the Galois group of $F$ over $\mathbb{Q}$ is cyclic of order $p^n$ for some prime number $p$.

11. Let $F \subset B \subset E$ be a chain of extensions such that $F \subset B$ is normal and $B \subset E$ is normal. Is it always true that $F \subset E$ is normal?

12. Find the Galois group of $(x^2 - 3)(x^2 + 1)(x^3 - 6)$ over $\mathbb{Q}$.

13. Find the Galois group of $x^4 + 3x + 5$ over $\mathbb{Q}$.

14. Let $p$ be a prime number. Prove that $n\sqrt{p}$ is constructible if and only if $n = 2^k$ for some $k$.

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15. Prove that any subfield of $\mathbb{Q}(\sqrt[n]{2})$ coincides with $\mathbb{Q}(d\sqrt[2]{2})$ for some divisor $d$ of $n$.

16. Prove that there exists a polynomial of degree 7 whose Galois group over $\mathbb{Q}$ is $\mathbb{Z}_7$.

17. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of odd prime degree $p$ solvable in radicals. Prove that the number of real roots of $f(x)$ equals $p$ or 1.

18. Let $f(x) \in \mathbb{F}_2[x]$ be an irreducible polynomial. Prove that $f(x)$ divides $x^{256} - x$ if and only if the degree of $f(x)$ is 1, 2, 4 or 8.

19. Suppose that the Galois group over $\mathbb{Q}$ of a polynomial $f(x) \in \mathbb{Q}[x]$ has odd order. Prove that all roots of $f(x)$ are real.

20. Find the Galois group of $x^6 - 8$ over $\mathbb{Q}$.