Solutions of homework problems.
Math 113
Vera Serganova

21 (7.7) Assume that $Z(G) \neq \{e\}$ but G is not abelian, then $|Z(G)| = p$ or q, and therefore $|G/Z(G)| = p$ or q. But then $G/Z(G)$ is cyclic. By Theorem 7.38, G is abelian. Contradiction.

11 (7.8) Define $f : \mathbb{R}^* \to \mathbb{R}^{**}$ by $f(x) = |x|$. Then f is a surjective homomorphism, and $\operatorname{Ker} f = \{1, -1\}$. By the first isomorphism theorem $\mathbb{R}^*/\{1,-1\}$ is isomorphic to $\mathbb{R}^{**}$.

21 (7.8)

a) The group of even permutations A_3 has three elements, hence it is abelian. The quotient S_3/A_3 has two elements and therefore it is also abelian. Thus S_3 is metabelian.

b) It is sufficient to show that if G is metabelian and $f : G \to H$ is a surjective homomorphism, then H is metabelian. Since G is metabelian, it has a normal abelian subgroup N such that G/N is abelain. Since f is surjective, $f(N)$ is an abelain normal subgroup of H. Define $\bar{f} : G/N \to H/f(N)$ by $\bar{f}(gN) = f(g)f(N)$. It is well defined and surjective. Hence $H/f(N)$ is abelian, and N is metabelian.

c) Let K be a subgroup of a metabelian group G. Let N be a normal subgroup of G such that G/N is abelian. Then $K \cap N$ is a normal abelian subgroup of K, and $K/(K \cap N)$ is a subgroup of G/N, therefore $K/(K \cap N)$ is abelian. Hence K is metabelian.

28 (8.1) (a) If S_3 decomposes into direct product, it must be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$, which is impossible because S_3 is not abelian.

(b) Since $|D_4| = 8$, $D_4 = H \times K$ would imply $|H| = 4$, $|K| = 2$. But then H and K are abelian, which means that D_4 is abelian. Contradiction.

(c) Every two non-trivial subgroups of $\mathbb{Z}$ are cyclic, so they coincide with (m) and (n) for some $m, n \in \mathbb{Z}$. Then they have a non-trivial intersection (k), where k is the least common multiple of m and n. Therefore $\mathbb{Z}$ can not be a direct product of these groups. Hence $\mathbb{Z}$ is indecomposable.

8 (8.2)

$$G(2) = \left\{ \left[\frac{m}{2^n} \right] \mid 0 \leq m < 2^n, (m,2) = 1 \right\},$$
$$G(p) = \left\{ \left[\frac{m}{p^n} \right] \mid 0 \leq m < p^n, (m,p) = 1 \right\}.$$

20 (8.2) Let $G = \mathbb{Z}_3 \oplus \mathbb{Z}$, S be the set of all elements of infinite order. Then $S = \{(a,b) \mid b \neq 0\}$, and $S \cup \{(0,0)\}$ is not a subgroup, for example $(1,2), (0,-2) \in S$ but $(1,0) = (1,2) + (0,-2) \notin S$.

17 (8.2)

a) Using the fundamental theorem

$$G \cong \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}, \quad H \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_l},$$

where $m_1 \geq \cdots \geq m_k$, $n_1 \geq \cdots \geq n_l$ are elementary divisors. Then

$$G \oplus G \cong \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k} \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}, \quad H \oplus H \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_l} \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_l},$$

and the elementary divisors of $G \oplus G$ are $m_1, m_1, \ldots, m_k, m_k$, the elementary divisors of $H \oplus H$ are $n_1, n_1, \ldots, n_l, n_l$. Since $G \oplus G$ and $H \oplus H$ are isomorphic, they have the same elementary divisors. Therefore $k = l$, and $m_i = n_i$. But that implies that G and H are isomorphic.

b) The same idea. If the elementary divisors of $G \oplus K$ are the same as the elementary divisors of $H \oplus K$, then the groups G and H have the same elementary divisors.

6 (8.3) In all cases the group is cyclic by Corollary 8.18.

7 (8.3) Using the third Sylow theorem, one can show that the group has only one Sylow p-subgroup, where $p = 7$ in (a), 5 in (b), 11 in (c) and 17 in (d). If the Sylow p-subgroup is unique, it is normal.

13 (8.3) Assume that $|G| = p_1^{n_1} \ldots p_r^{n_r}$, denote by K_i the Sylow p_i-subgroup. First note that $K_i \cap K_j = \{e\}$ if $i \neq j$. Therefore $ab = ba$ for any $a \in K_i$, $b \in K_j$. Define a map $f : K_1 \times \cdots \times K_r \to G$ by $f(a_1, \ldots, a_r) = a_1 \ldots a_r$. The above property implies that f is a homomorphism. To check that f is injective, note that $K_1 \cap (K_2 \ldots K_r) = \{e\}$, $K_2 \cap (K_3 \ldots K_r) = \{e\}, \ldots K_{r-1} \cap K_r = \{e\}$. Therefore $f(a_1, \ldots, a_r) = e$ implies $a_1^{-1} = a_2 \ldots a_n \in K_1 \cap (K_2 \ldots K_r)$, hence $a_1 = e$, similarly $a_2 = \cdots = a_r = e$. Finally, since $|G| = |K_1| \ldots |K_r|$, f is an isomorphism.

14 (8.3) Let $|G| = p^a m$, $|N| = p^b k$ for some m, k such that $(p, m) = (p, k) = 1$. Since $|N|$ divides $|G|$, $b \leq a$. Use the isomorphism $\frac{K}{K \cap N} \cong \frac{KN}{N}$, hence $\frac{|K|}{|K \cap N|} = \frac{|KN|}{|N|}$. Note that since K is a subgroup of KN, $p^a || KN|$, therefore $\frac{|K|}{|K \cap N|} = p^{a-b} m'$ for some m' which is not divisible by p. On the other hand, $\frac{|K|}{|K \cap N|}$ is a power of p, hence $\frac{|K|}{|K \cap N|} = p^{a-b}$. Therefore $|K \cap N| = p^b$, and $K \cap N$ is a Sylow subgroup of N.

5 (8.4) $<(123)>, <(124)>, <(234)>, <(134)>$.

9 (8.4) Let f be an automorphism of G. If $C = \{gag^{-1} \mid g \in G\}$, then

$$f(C) = \left\{ f\left(gag^{-1}\right) \mid g \in G \right\} = \left\{ f(g) f(a) (f(g))^{-1} \mid g \in G \right\} = \left\{ g f(a) g^{-1} \mid g \in G \right\}.$$

10 (8.4) Let $a, b \in H$, and C_1, C_2 be the conjugacy classes of a and b respectively. Then the conjugacy class of ab is

$$C = \left\{ gabg^{-1} = gag^{-1} gbg^{-1} | g \in G \right\} \subset C_1 C_2.$$

But C_1 and C_2 are finite sets, so $C_1 C_2$ is finite and C is finite as a subset of a finite set. Hence $ab \in H$. Clearly, $e \in H$, because its conjugacy class has only one element. Finally, $a^{-1} \in H$ because its conjugacy class is C_1^{-1}. Therefore H is a subgroup of G.