Solutions for the sample midterm.

1. No solutions. Indeed, $x^2 = 3 + 2007n$ for some integer n. Then 3 divides x^2, hence 3 divides x. Therefore 9 divides x^2. However, 9 does not divide $3 + 2007n$.

2. Note that 19 is a prime number. By Fermat's little theorem, $3^{18} \equiv 1 \mod 19$. Since $198 = 18 \cdot 11$. If $3^{198} = 1$ in $\mathbb{Z}_{19}$.

3.(a) R is commutative because the multiplication table is symmetric.

(b) R is a ring with identity element, $d = 1_R$.

(c) R is not a field because b does not have inverse.

4. All units in the ring $\mathbb{Z}_9$ are 1,2,4,5,7,8, their inverses are 1,5,7,2,4,8 respectively.

5. The units of $\mathbb{Z}_3 \times \mathbb{Z}_3$ are (1,1),(1,2),(2,1),(2,2). Therefore $\mathbb{Z}_3 \times \mathbb{Z}_3$ has less units than $\mathbb{Z}_9$. Therefore $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_9$ are not isomorphic.

6.

a) Not true. Counterexample:

$$f : \mathbb{Z}_3 \times \mathbb{Z}_3 \to \mathbb{Z}_3,\ f(a, b) = b.$$

b) True. Let $f : R \to S$ be an injective homomorphism of rings and S be an integral domain. Let $a, b \in R$, $ab = 0_R$. Then $f(a) f(b) = 0_S$. Since S is an integral domain, $f(a) = 0_S$ or $f(b) = 0_S$. But f is injective, therefore $f(a) = 0_S$ implies $a = 0_R$, $f(b) = 0_S$ implies $b = 0_R$. Thus one of a, b is 0_R.