1)

7.7

#13 $\mathbb{R}^*/\mathbb{R}^{**}$ has two cosets $\mathbb{R}^{**}$ and $-\mathbb{R}^{**}$ (Positive and negative real numbers).

Thus $|\mathbb{R}^*/\mathbb{R}^{**}| = 2$, hence isomorphic to $\mathbb{Z}_2$.

#21 If $Z(G) \neq e$ or G then $G/Z(G)$ and $Z(G)$ are both of prime order, hence cyclic.

But then by Theorem 7.38, G is abelian.

So $Z(G) = G$.

#22 Let $Ny_1, \ldots, Ny_n$ be generators of G/N and $x_1 \ldots x_m$ be generators of N. Every $g \in G$ belongs to $N\langle y_1, \ldots, y_n\rangle$, so $g = n\, y_{i_1}^{\pm 1} \ldots y_{i_k}^{\pm 1}$.

But $n = x_{j_1}^{\pm 1} \ldots x_{j_e}^{\pm 1}$. So

$g = x_{j_1}^{\pm 1} \ldots x_{j_e}^{\pm 1}\, y_{i_1}^{\pm 1} \ldots y_{i_k}^{\pm 1} \in \langle x_1 \ldots x_m, y_1, \ldots, y_m\rangle$

2)

#23

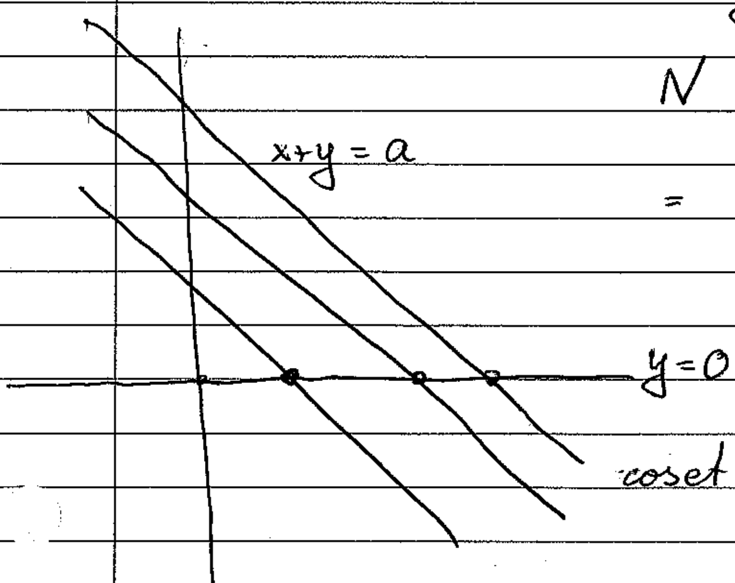
(a) The set of all commutaters $S = \{aba^{-1}b^{-1}\}$ is invariant under conjugation $gSg^{-1} = S$. Hence $G' = \langle S \rangle$ is also invariant under conjugation. Hence G' is normal.

(b) Theorem 7.37. For any $a, b \in G$ $aba^{-1}b^{-1} \in G'$. So G/G' is abelian.

#24 (a) $(x,-x) + (y,-y) = (x+y, -(x+y))$

$-(x,-x) = (-x, -(-x))$

(b) G/N is isomorphic to $\mathbb{R}$. To prove this note that every coset intersects the line $y=0$ at one point



$N + (x,0) + N + (x',0) =$

$= N + (x+x')$

isomorphism

(homomorphism and bijection

#26.

Let $\Phi_g \in \text{Inn } G$ be defined by

$\Phi_g(a) = g a g^{-1}$

Let $f: G \longrightarrow \text{Inn } G$ be defined

by $f(g) = \Phi_g$. Then

$\Phi_{gh}(a) = gh a \cdot (gh)^{-1} = g(hah^{-1})g^{-1} = \Phi_g \circ \Phi_h(a)$.

f is a surjective homomorphism.

$\text{Ker } f = \{ g \in G \mid \underset{\substack{\Updownarrow \\ ga = ag}}{gag^{-1} = a} \quad \forall a \in G \} = Z(G)$.

Now the statement follows from

the first isomorphism theorem.

4)

7.8.

\# 8

f is well defined. Indeed $x \equiv y \pmod{n}$

$\Rightarrow n \mid x-y \Rightarrow k \mid x-y$

$f([x]_n + [y]_n) = f([x+y]_n) = [x+y]_k =$

$= [x]_k + [y]_k = f([x]_n) + f([y]_n)$

$\operatorname{Ker} f = \{0, k, 2k, \ldots, (p-1)k\}$ where $p = \frac{n}{k}$
cyclic group of order p.

\#10 (a) $K = \{a \in G \mid a^2 = e\}$

$a \in K, b \in K \Rightarrow (ab)^2 = abab = a^2b^2 = e \Rightarrow ab \in K$
$a \in K, (a^{-1})^2 = (a^2)^{-1} = e \Rightarrow a^{-1} \in K$

(b) $c = a^2, d = b^2 \Rightarrow cd = (ab)^2$
$c = a^2, \Rightarrow c^{-1} = (a^{-1})^2 \Rightarrow c^{-1} \in H$

(c) Let $f: G \longrightarrow H$ be defined by

$f(x) = x^2$ Then $f(xy) = (xy)^2 = x^2y^2 = f(x)f(y)$

By construction $\operatorname{Im} f = H$, $\operatorname{Ker} f = K$.

So $G/K \cong H$ by the first isomorphism theorem.

5)

#11. Let $f : \mathbb{R}^* \to \mathbb{R}^{**}$ given by $f(a) = |a|$. $\operatorname{Ker} f = \{\pm 1\}$, f is surjective. Use first isomorphism theorem.

#17.

(a) $$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+a' & b+b'+ac' \\ 0 & 1 & c+c' \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a & -b+ac \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$$

(b) $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & a & 1 \end{pmatrix} \in Z(G)$ if and only if $b + b' + ac' = b + b' + a'c$ for any a', b', c'. i.e $ac' = a'c$ $\forall a', c'$ $\Rightarrow a = c = 0$

$C = Z(G) = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, b \in \mathbb{Q} \right\}$

(c) $f : G \to \mathbb{Q} \times \mathbb{Q}$

$f\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 0 \end{pmatrix} = (a, c)$ (homomorphism by (a))

$\operatorname{Ker} f = C$. Use first isomorphism theorem.

6)

#21 (a) $A_3 = \langle (123) \rangle \cong \mathbb{Z}_3$

$|S_3/A_3| = 2 \Rightarrow S_3/A_3 \cong \mathbb{Z}_2$

(b) $f(N)$ is abelian

N is an abelian normal subgroup of G, G/N is also abelian.

~~Net~~ $f: G \to H$ be a homomorphism, and

$\bar{f}: G/N \to f(G)/f(N)$ be

defined by $\bar{f}(Nx) = f(N)f(x)$

$\bar{f}$ is surjective. So $f(N)$ is

abelian and $f(G)/f(N)$ is abelian

So $f(G)$ is metabelian.

(c) Let $K \subset G$ be a subgroup and

G be metabelian. So there is $N \subset G$

normal abelian such that G/N is abelian.

Put $N' = K \cap N$. Then N' is normal

abelian in K. and $\frac{K}{N'} \cong \frac{KN}{N}$ (second iso. thm)

But $\frac{KN}{N}$ is a subgroup of G/N

Hence it is abelian. So K is metabelian.