13. \( \mathbb{R}^* / \mathbb{R}^{**} \) has two cosets \( \mathbb{R}^{**} \) and \( -\mathbb{R}^{**} \) (Positive and negative real numbers).

Thus \( |\mathbb{R}^*/\mathbb{R}^{**}| = 2 \), hence isomorphic to \( \mathbb{Z}_2 \).

21. If \( Z(G) \neq e \) in \( G \) then \( G/\mathbb{Z}(G) \) and \( \mathbb{Z}(G) \) are both of prime order, hence cyclic.

But then by Theorem 7.38, \( G \) is abelian. So \( \mathbb{Z}(G) = G \).

22. Let \( N \langle y_1, \ldots, y_n \rangle \) be generators of \( G/N \) and \( x_1, \ldots, x_m \) be generators of \( N \). Every \( g \in G \) belongs to \( N \langle y_1, \ldots, y_n \rangle \), so \( g = n \ y_{i_1}^{\pm 1} \ldots y_{i_k}^{\pm 1} \).

But \( n = x_{j_1}^{\pm 1} \ldots x_{j_e}^{\pm 1} \).

So \( g = x_{j_1}^{\pm 1} \ldots x_{j_e}^{\pm 1} y_{i_1}^{\pm 1} \ldots y_{i_k}^{\pm 1} \in \langle x_1, \ldots, x_m, y_1, \ldots, y_m \rangle \).
#23

(a) The set of all commutators $S=\{aba^{-1}b^{-1}\}$ is invariant under conjugation $gSg^{-1}=S$. Hence $G^\prime = \langle S \rangle$ is also invariant under conjugation. Hence $G^\prime$ is normal.

(b) Theorem 7.37. For any $a, b \in G$, $aba^{-1}b^{-1} \in G^\prime$. So $G/G^\prime$ is abelian.

#24

(a) $(x, -x) + (y, -y) = (x+y, -(x+y))$
    $-(x, -x) = (-x, -(-x))$

(b) $G/N$ is isomorphic to $\mathbb{R}$. To prove this, note that every coset intersects the line $y = 0$ at one point. For $xy = a$, $N + (x, 0) + N + (y, 0) = N + (x+y, 0)$, which is an isomorphism between cosets and bijection.
26. Let $\Phi_g \in \text{Inn } G$ be defined by

$$\Phi_g(a) = g \cdot a \cdot g^{-1}$$

Let $f : G \rightarrow \text{Inn } G$ be defined by $f(g) = \Phi_g$. Then

$$\Phi_{gh}(a) = gh \cdot a \cdot (gh)^{-1} = g \cdot (h \cdot a \cdot h^{-1}) \cdot g^{-1} = \Phi_g \circ \Phi_h(a)$$

$f$ is a surjective homomorphism.

Let $F = \{ g \in G \mid g \cdot a \cdot g^{-1} = a \forall a \in G \}, \quad F = Z(G)$.

$$g a = a g$$

Now the statement follows from

the first isomorphism theorem.
7.8.

# 8

$f$ is well defined. Indeed $x \equiv y \pmod{n} \Rightarrow n \mid x - y \Rightarrow k \mid x - y$

$f([x]_n + [y]_n) = f([x+y]_n) = [x+y]_k = [x]_k + [y]_k = f([x]_n) + f([y]_n)$

$\ker f = \{ 0, k, 2k, \ldots, (p-1)k \}$ where $p = \frac{n}{k}$

cyclic group of order $p$.

# 10 (a) $K = \{ a \in G \mid a^2 = e \}$

$a, b \in K \Rightarrow (ab)^2 = abab = a^2b^2 = e \Rightarrow ab \in K$

$a \in K, (a^{-1})^2 = (a^2)^{-1} = e \Rightarrow a^{-1} \in K$

(b) $c = a^2, d = b^2 \Rightarrow cd = (ab)^2$

$c = a^2, \Rightarrow c^{-1} = (a^{-1})^2 \Rightarrow a^{-1} \in H$

(c) Let $f : G \rightarrow H$ be defined by

$f(x) = x^2$  Then $f(xy) = (xy)^2 = x^2y^2 = f(x)f(y)$

By construction $\text{Im} f = H$, $\ker f = K$

So $G/H \cong H$ by the first isomorphism theorem.
#11. Let \( f : \mathbb{R}^* \to \mathbb{R}^* \) given by \( f(a) = |a| \). \( \ker f = \{ \pm 1 \} \). \( f \) is surjective. Use first isomorphism theorem.

#17.

(a) \[
\begin{pmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & a' & b' \\
0 & 1 & c' \\
0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & a + b + ac \\
0 & 1 & c + c' \\
0 & 0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1
\end{pmatrix}
^{-1}
= \begin{pmatrix}
a & -b & -ac \\
0 & 1 & -c \\
0 & 0 & 1
\end{pmatrix}
\]

(b) \( \begin{pmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1
\end{pmatrix} \in Z(G) \) if and only if \( b + b' + ac = b + b' + a'c \) for any \( a, b, c \). i.e. \( ac = a'c \) \( \# a, c' \Rightarrow a = c = 0 \)

\( C = Z(G) = \{ \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \mid b \in \mathbb{Q} \} \)

(c) \( f : G \to \mathbb{Q} \times \mathbb{Q} \)

\[ f \left( \begin{pmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1
\end{pmatrix} \right) = (a, c) \quad \text{(homomorphism)} \] by (a)

\( \ker f = C. \) Use first isomorphism theorem.
6) 

(a) \( A_3 = \langle (123) \rangle \cong \mathbb{Z}_3 \)

\[ S_3 / A_3 = 2 \Rightarrow S_3 / A_3 \cong \mathbb{Z}_2 \]

(b) \( f(N) \) is abelian

\( N \) is an abelian normal subgroup of \( G \), \( G/N \) is also abelian.

Let \( f : G \to H \) be a homomorphism, and

\[ \tilde{f} : G/N \to f(G)/f(N) \]

defined by \( \tilde{f}(Nx) = f(N)f(x) \).

\( \tilde{f} \) is surjective. So \( f(N) \) is abelian and \( f(G)/f(N) \) is abelian.

So \( f(G) \) is metabelian.

(c) Let \( K \leq G \) be a subgroup and \( G \) be metabelian. So there is \( N \leq G \) normal abelian such that \( G/N \) is abelian.

Put \( N' = K \cap N \). Then \( N' \) is normal abelian in \( K \) and \( \frac{K}{N'} \cong \frac{KN}{N} \) (second iso. thm).

But \( \frac{KN}{N} \) is a subgroup of \( G/N \)

Hence it is abelian. So \( K \) is metabelian.