1. List all units in the ring $\mathbb{Z}_9$. For each unit $a \in \mathbb{Z}_9$ find its multiplicative inverse $a^{-1}$.

2. Show that the ring $\mathbb{Z}_9$ is not isomorphic to the ring $\mathbb{Z}_3 \times \mathbb{Z}_3$.

3. Let $F$ be a field, $c \in F$, and $F[x]$ be the polynomial ring. Define the map $f : F[x] \rightarrow F$ by the formula

$$f(a_0 + a_1 x + \cdots + a_n x^n) = a_0 + a_1 c + \cdots + a_n c^n.$$ 

a) Show that $f$ is a surjective homomorphism.

b) Show that the kernel of $f$ is the principal ideal $(x - c)$.

c) Show that $F[x]/(x - c)$ is isomorphic to $F$.

5. Let $f : \mathbb{Z}_{10} \rightarrow \mathbb{Z}_{10}$ be the map defined by the formula $f(x) = 6x$.

a) Is $f$ a homomorphism?

b) Is $f$ an isomorphism?

Justify your answers.

6. Determine which of the following quotient rings are fields. Explain your answer.

a) $\mathbb{Q}[x]/(x^3 + x + 1)$;

b) $\mathbb{Z}_{11}[x]/(x^3 + x + 1)$;

c) $\mathbb{R}[x]/(x^3 + x + 1)$.

7. Let $K = \mathbb{Z}[x]/(x^3 + x + 1)$.

a) Is $K$ an integral domain?

b) Is $K$ a field?

8. Find all solutions for the equation

$$x^3 = x$$
in $\mathbb{Z}_{10}$.

9. Using Fermat’s little theorem prove that $29 \mid 1000^{840} - 1$.

10. Let $R$ be a subring of $\mathbb{Z}$. Prove that if $R$ contains 13 and 1000, then $R = \mathbb{Z}$.

11. Determine which of the following rings are integral domains. Explain your answers.

a) $\mathbb{Q} \times \mathbb{R}$;

b) $\mathbb{Z}[x]$;

c) $\mathbb{Z}_{10}[x]$.

12. Prove that $\mathbb{Z}$ and $\mathbb{Z}[x]$ are not isomorphic.

13. Let $f : F \rightarrow R$ be a homomorphism of rings. Show that if $F$ is a field, then $f$ is either injective or the zero map.

14. List all irreducible polynomials of degree 4 in $\mathbb{Z}_2[x]$. Explain why your list is complete and why all polynomials in your list are irreducible.
15. Let \( R \) be a ring of four elements \( \{a, b, c, d\} \) with addition and multiplication given by the tables

\[
\begin{array}{cccc}
+ & a & b & c & d \\
\hline
a & a & b & c & d \\
b & b & a & d & c \\
c & c & d & a & b \\
d & d & c & b & a \\
\end{array}
\quad \begin{array}{cccc}
\cdot & a & b & c & d \\
\hline
a & a & a & a & a \\
b & a & b & a & b \\
c & a & a & c & c \\
d & a & b & c & d \\
\end{array}
\]

(a) Is \( R \) commutative?
(b) Is \( R \) a ring with identity element?
(c) Is \( R \) a field?