Solutions of homework problems.
Math 113

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25 (7.4) Let $f_1(a) = c^{-1}ac$ and $f_2(a) = d^{-1}ad$. Then $f_1^{-1}(a) = cac^{-1}$ and hence $f_1^{-1} \in$Inn G, $f_1 \circ f_2(a) = (dc)^{-1} a(dc)$, and hence $f_1 \circ f_2 \in$Inn G.

27 (7.4) We have to show that $\mathbb{Q}$ is not cyclic. Indeed, assume that $\mathbb{Q}$ is cyclic, let $c \in \mathbb{Q}$ be a generator. Then for any $x \in Q$, there exists $n \in \mathbb{Z}$ such that $x = nc$. Take $x = \frac{c}{2}$ and obtain contradiction.

43 (7.4) Let $f \in \operatorname{Aut} \mathbb{Z}_n$. Since 1 is a generator of $\mathbb{Z}_n$, $u = f(1)$ is a generator of $\mathbb{Z}_n$, and therefore $(u, n) = 1$. Then for any $m \in \mathbb{Z}_n$

$$f(m) = f(m1) = mf(1) = mu.$$

Therefore any automorphism f is equal to $f_u : \mathbb{Z}_n \to \mathbb{Z}_n$ defined by the formula

$$f_u(x) = xu,$$

with $u \in U_n$. Moreover

$$f_u \circ f_v(x) = xuv = x(vu).$$

Thus, $f_u \circ f_v = f_{uv}$, and the map $f_u \mapsto u$ defines an isomorphism $\operatorname{Aut} \mathbb{Z}_n \to U_n$.

12 (7.5) If G is not cyclic, then $|a| \neq 25$ for $a \in G$. Since $|a|$ divides $|G| = 25$, and $|a| \neq 1$ if $a \neq e$, $|a| = 5$.

18 (7.5) Let $[G : H] = m$, $[H : K] = n$. Then G is a disjoint union of H-cosets

$$a_1 H \cup \cdots \cup a_n H,$$

and H is a disjoint union of K-cosets

$$b_1 K \cup \cdots \cup b_m K.$$

Then the cosets $a_i b_j K$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$ cover the whole G. We want to show that none of them coincide. Indeed, assume

$$a_i b_j K = a_p b_q K.$$

Then

$$a_p^{-1} a_i \in b_q K b_j^{-1} \in H,$$

and therefore $a_i = a_p$. Then $b_j K = b_q K$ and $b_j = b_q$.

Thus, the number of K-cosets in G is mn.

22 (7.5) Since the number of all subsets of G containing e is equal to 2^{n-1}, any subset S of G such that $e \in S$ is a subgroup. In particular for any $a \in G$, $a \neq e$ the subset $\{e, a\}$ is a subgroup. Therefore $|a| = 2$ for any $a \in G$, $a \neq e$. Assume that $|G| > 2$. Then choose a 3-element set $S = \{e, a, b\}$. We have shown that S is a subgroup, but the order of a does not divide $|S|$. Contradiction. Thus, we obtain $|G| = 1, 2$, and the statement follows.

24 (7.5) Suppose that the statement is false. Then G is not cyclic, and none of its elements has order 3. Then every $a \in G$, $a \neq e$ must have order 11. Let $H =< a >$ be the cyclic subgroup generated by a. Then G has 3 cosets H, Hb and Hb^2, where

b is some element which does not belong to H. Then $b^3 \in H$, and therefore $b^3 = a^k$ for some $k < 11$. If $k = 0$ we are done, if $k > 0$ let $x = b^{11}$, and then $x^3 = b^{33} = e$. On the other hand,

$$b^{11} = \left(b^3\right)^3 b^2 = a^{3k} b^2 \in Hb^2,$$

and hence $x \neq e$. Thus $|x| = 3$.

7 (7.6) Just check that

$$(b, c)^{-1} (a, e) (b, c) = \left(b^{-1}ab, c^{-1}ec\right) = \left(b^{-1}ab, e\right) \in G^*.$$

16 (7.6) If $x \in G$, $y \in K \cap N$, then $x^{-1}yx \in K$ and $x^{-1}yx \in N$ since both K and N are normal.

17 (7.6) If $x \in K$, $y \in N \cap K$, then $x^{-1}yx \in K$ since K is a subgroup, and $x^{-1}yx \in N$ since N is a normal subgroup.

18 (7.6) If $x_1, x_2 \in N$, $y_1, y_2 \in K$, then

$$x_1 y_1 x_2 y_2 = x_1 \left(y_1 x_2 y_1^{-1}\right) y_1 y_2 \in NK,$$

and

$$(x_1 y_1)^{-1} = y_1^{-1} x_1^{-1} = \left(y_1^{-1} x_1^{-1} y_1\right) y_1^{-1} \in NK.$$

That shows that NK is a subgroup.

If K is also normal for any $g \in G$

$$g^{-1} x_1 y_1 g = \left(g^{-1} x_1 g\right) \left(g^{-1} y_1 g\right) \in NK, \text{ and } NK \text{ is normal.}$$

21 (7.6) Any $x \in H$ can be written as $f(y)$ for some $y \in G$. Any $b \in f(N)$ can be written as $f(a)$ for some $a \in N$. Then

$$x^{-1}bx = f\left(y^{-1}\right) f(a) f(y) = f\left(y^{-1}ay\right) \in f(N).$$

Hence $f(N)$ is normal.

24 (7.6) Note that $x^{-1}Hx$ is a subgroup of G of the same order as H. If H is a unique subgroup of order n, $x^{-1}Hx = H$.