Solutions of homework problems.
Math 113
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25(3.2) Use the identity $a+a=(a+a)^2=a^2+a^2+a^2+a^2=a+a+a+a$, which implies that $a+a=0_R$, or $a=-a$. Now write $a+b=(a+b)^2=a^2+ab+ba+b^2=a+ab+ba+b$. Hence $ab+ba=0_R$ or $ab=-ba=ba$.

31(3.2) $n\cdot 1=n\neq 0$. Hence characteristic of $\mathbb{Z}$ is zero. In $\mathbb{Z}_n n\cdot 1=n=0$, and for any $m<n$, $m\cdot 1=m\neq 0$, hence characteristic of $\mathbb{Z}_n$ is n. In $\mathbb{Z}_4\times\mathbb{Z}_6$

$$n(1,1)=0$$

if and only if $4|n$ and $6|n$. The smallest such n is 12. Hence the characteristic of $\mathbb{Z}_4\times\mathbb{Z}_6$ is 12.

32(3.2) If R is a finite set, then $m1_R=k1_R$ for some $m\neq k$. Therefore $(m-k)\,1_R=0_R$. The smallest positive n such that $n1_R=0_R$ is the characteristic of R.

33 (3.2)

$$na=n\,(1_Ra)=(n1_R)\,a=0_Ra=0_R.$$

Assume that n is composite, let $n=pq$. Then $p1_R\neq 0_R$, $q1_R\neq 0_R$, but

$$(p1_R)\,(q1_R)=(pq)\,1_R=0_R.$$

Hence R is not an integral domain. Contradiction.

17(3.3) One can describe S as the subset of $\mathbb{Z}_{28}$ of elements of the form $4k$ for some $k\in\mathbb{Z}_{28}$. Since $4k-4l=4\,(k-l)$, and $4k4l=4\,(4kl)$, S is a subring of $\mathbb{Z}_{28}$. To check that f is well defined note that $x\equiv y \mod 7$ implies $8x\equiv 8y \mod 28$. Next check that $f:\mathbb{Z}_7\to\mathbb{Z}_{28}$ preserves addition and multiplication

$$f\left([x]_7+[y]_7\right)=[8\,(x+y)]_{28}=[8x+8y]_{28}=[8x]_{28}+[8y]_{28}=f\left([x]_7\right)+f\left([y]_7\right),$$

$$f\left([x]_7\,[y]_7\right)=[8\,(xy)]_{28}=[8x8y]_{28}=f\left([x]_7\right)f\left([y]_7\right),$$

to show the second identity we use $8^2\equiv 8 \mod 28$. Finally, to check that f is an isomorphism it is sufficient to show that $\operatorname{Ker} f=\{0\}$. Indeed, let $f\left([x]_7\right)=0$, then $8x\equiv 0 \mod 28$. In other words, $28|8x$. Since $(7,8)=1$, $7|x$. Hence $[x]_7=0$.

23(3.3)

$$f\begin{pmatrix} a+a' & 0\\ b+b' & c+c'\end{pmatrix}=a+a'=f\begin{pmatrix} a & 0\\ b & c\end{pmatrix}+f\begin{pmatrix} a' & 0\\ b' & c'\end{pmatrix},$$

$$f\begin{pmatrix} aa' & 0\\ ba'+cb' & cc'\end{pmatrix}=aa'=f\begin{pmatrix} a & 0\\ b & c\end{pmatrix}f\begin{pmatrix} a' & 0\\ b' & c'\end{pmatrix}.$$

The map is surjective since $a=f\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix}$for any $a\in\mathbb{R}$. The map is not injective, since $f\begin{pmatrix} 0 & 0\\ b & c\end{pmatrix}=0$ for all b and c.

33(3.3) (a) E does not have identity, but $\mathbb{Z}$ has one. (b) The first ring is commutative. The second ring is not. (c) Different number of elements. (d) if there is an isomorphism $f:\mathbb{R}\to\mathbb{Q}$, then $x=f\left(\sqrt{2}\right)$ satisfies the equation $x^2=2$. But such equation does not have solutions in $\mathbb{Q}$. (e) The second ring is an integral domain , the first ring is not. (f) If $f\colon \mathbb{Z}_4 \times \mathbb{Z}_4 \to \mathbb{Z}_{16}$ is an isomorphism, then $f\left([1]_4, [1]_4\right) = [1]_{16}$, $f\left([0]_4, [0]_4\right) = [0]_{16}$. But

$$f\left([0]_4, [0]_4\right) = f\left(4\left([1]_4, [1]_4\right)\right) = 4f\left([1]_{16}\right) = [4]_{16} \neq 0.$$

40(3.3) Assume that $(m, n) = d > 1$. Let $k = \frac{mn}{d}$. Then $m|k$ and $n|k$. Assume that there is an isomorphism $f : \mathbb{Z}_m \times \mathbb{Z}_n \to \mathbb{Z}_{mn}$. Then

$$f\left([1]_m, [1]_n\right) = [1]_{mn},$$
$$f\left([0]_m, [0]_n\right) = f\left(k\left([1]_m, [1]_n\right)\right) = kf\left([1]_{mn}\right) = [k]_{mn} \neq [0]_{mn}.$$

3(4.1) There are eight polynomials of degree 3 in $\mathbb{Z}_2[x]$: x^3, x^3+x^2, x^3+x, x^3+1, x^3+x^2+1, x^3+x+1, x^3+x^2+x, x^3+x^2+x+1.

There are 27 polynomials of degree less than 3 in $\mathbb{Z}_3[x]$. Here they are

$$0, 1, 2, x, x+1, x+2, 2x, 2x+1, 2x+2$$
$$x^2, x^2+1, x^2+2, x^2+x, x^2+x+1, x^2+x+2, x^2+2x, x^2+2x+1, x^2+2x+2$$
$$2x^2, 2x^2+1, 2x^2+2, 2x^2+x, 2x^2+x+1, 2x^2+x+2, 2x^2+2x, 2x^2+2x+1, 2x^2+2x+2$$