From similarity over $\mathbb{C}$ to $\mathbb{Q}$

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This is a problem from Berkeley's preliminary exam.

Theorem 1. *Let A, B be $n \times n$ matrices with coefficients in $\mathbb{Q}$. For any field extension K of $\mathbb{Q}$, we say that A and B are similar over K if $A = PBP^{-1}$ for some $n \times n$ invertible matrix P with coefficients in K. Prove that A and B are similar over $\mathbb{Q}$ if and only if they are similar over $\mathbb{C}$.*

Proof. The *only if* part is trivial, and we will show the *if* part. Let $P = (x_{ij})$. The condition that A and B are similar over $\mathbb{C}$ is equivalent to the statement that the system of equation

$$AP = PB, \quad \det(P) \neq 0$$

has a solution in $(x_{ij}) \in \mathbb{C}^{n^2} \simeq M_{n\times n}(\mathbb{C})$. We need to show that the system has a solution in $\mathbb{Q}^{n^2}$.

Let $V \subset \mathbb{Q}^{n^2}$ be a space of solution of $AP = PB$ and let $W \subset \mathbb{C}^{n^2}$ be a space of solution of the same equation. Then we have a natural isomorphism $W \simeq V \otimes_{\mathbb{Q}} \mathbb{C}$. (The system of equation is just a linear equation over $\mathbb{Q}$). Since $W \neq 0$, $V \neq 0$ and we have a rational solution.

Now we have to consider the condition $\det(P) \neq 0$. Take a basis of V over $\mathbb{Q}$, and identify V with $\mathbb{Q}^m$ for some m. Then this basis also gives an identification $W = \mathbb{C}^m$. Now the restriction of $\det(P)$ to V becomes a polynomial $f(y_1, \ldots, y_m)$ of m variables with rational coefficients via the identification. By the condition, f is not identically zero over $\mathbb{C}$, so it is not the zero polynomial. Hence, we can find an element of $\mathbb{Q}^m$ at which f is nonzero, and this element gives the desired solution in $\mathbb{Q}^{n^2}$ of the equation. □

As you can see in the proof, this theorem holds for any arbitrary field extension K/F of characteristic zero. More precisely, if A, B are $n \times n$ matrices over a characteristic zero field F which are similar over a field K, then it is also similar over F.