the tvc and metric structures

second vaught’s conjecture workshop
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Overview

- Vaught’s conjecture has numerous model-theoretic, dynamical, and descriptive analogs.
- The $\mathcal{L}_{\omega_1\omega}$-Vaught conjecture and the topological Vaught conjecture are connected by López-Escobar’s theorem, a cornerstone result linking countable model theory, descriptive set theory, and logic.
- This talk will feature a generalization of Lopez-Escobar’s theorem for metric structures.
- As a consequence we will obtain a new model-theoretic characterization of the topological Vaught conjecture.
The space of countable structures

Definition
If $\mathcal{L}$ is a countable relational language with symbols $R_i$ of arity $n_i$, then we define the space of countable $\mathcal{L}$-structures

$$\text{Mod}(\mathcal{L}) = \prod \mathcal{P}(\mathbb{N}^{n_i}).$$

Recall
For $\mathcal{L}$ a language, $\mathcal{L}_{\omega_1\omega}$ denotes the extension of first-order logic in which countable conjunctions and disjunctions are allowed.
(Note: We require formulas to have finitely many free variables.)

$L_{\omega_1\omega}$-VC, Vaught’s conjecture for $\mathcal{L}_{\omega_1\omega}$
For any sentence $\phi$ of $\mathcal{L}_{\omega_1\omega}$, the subset $\text{Mod}(\phi) \subset \text{Mod}(\mathcal{L})$ consisting of the models of $\phi$ meets either countably many or perfectly many isomorphism classes.
The logic action

Recall
The Polish group $S_\infty$ of permutations of $\mathbb{N}$ acts naturally on $\text{Mod}(\mathcal{L})$ by shifting the subsets of $\mathbb{N}^n_i$; this is called the logic action.

Remark
The logic action makes $\text{Mod}(\mathcal{L})$ into an $S_\infty$-space, and its orbits are precisely the isomorphism classes.

This action plays a key role in descriptive set theory.

Proposition (Becker–Kechris)
If $\mathcal{L}$ contains symbols of arbitrarily large arity, then $\text{Mod}(\mathcal{L})$ is a universal Borel $S_\infty$-space. That is for any Borel action $S_\infty \curvearrowright X$ there is a Borel $S_\infty$-embedding $i: X \hookrightarrow \text{Mod}(\mathcal{L})$. 
Dynamical Vaught variants

The role of the logic action leads one to the dynamical Vaught conjectures:

\[ \text{TVC}(S_\infty), \text{ the topological Vaught conjecture for } S_\infty \]
Any standard Borel \( S_\infty \)-space has countably many or perfectly many orbits.

\[ \text{TVC}(G), \text{ for } G \text{ a Polish group} \]
Any standard Borel \( G \)-space has countably many or perfectly many orbits.

\[ \text{TVC} \]
\[ \text{TVC}(G) \text{ holds for any polish } G \]
A connection between VC and TVC

The topological Vaught conjecture is stronger than the Vaught conjecture.

Proposition

\[ \text{TVC}(S_{\infty}) \text{ implies the } \mathcal{L}_{\omega_1 \omega}\text{-VC}. \]

Proof.

This is simply because \( \text{Mod}(\phi) \) is an example of a standard Borel \( S_{\infty} \)-space.

Question

Can we go beyond \( \mathcal{L}_{\omega_1 \omega} \)?
The key result: López-Escobar’s theorem

Question
Can we go beyond $\mathcal{L}_{\omega_1}\omega$?

López-Escobar’s theorem states that this is not necessary.

Theorem (Edgar George Kenneth López-Escobar)
If $X \subset \text{Mod}(\mathcal{L})$ is Borel and isomorphism-closed then there exists a sentence $\phi$ of $\mathcal{L}_{\omega_1}\omega$ such that $X = \text{Mod}(\phi)$.

This allows us to show that the Proposition was tight.

Corollary
$\text{TVC}(S_\infty)$ is equivalent to the $\mathcal{L}_{\omega_1}\omega$-VC.
Proof of the corollary

Corollary

\[ \text{TVC}(S_\infty) \text{ is equivalent to the } L_{\omega_1 \omega}-VC. \]

Proof.

(\(\Rightarrow\)) This was because \(\text{Mod}(\phi)\) is Borel.

(\(\Leftarrow\)) Let \(X\) be a standard Borel \(S_\infty\)-space.

- There exists \(L\) and a Borel \(S_\infty\)-embedding \(i: X \hookrightarrow \text{Mod}(L)\). Note that \(i(X)\) is Borel and isomorphism-closed.
- By López-Escobar’s theorem there exists a sentence \(\phi\) of \(L_{\omega_1 \omega}\) such that \(i(X) = \text{Mod}(\phi)\).
- By the \(L_{\omega_1 \omega}\)-VC, the image \(i(X)\) has countably or perfectly many isomorphism types.
- Hence \(X\) has countably many or perfectly many orbits. \(\square\)
Idea of Vaught’s proof of López-Escobar’s theorem

If $X \subset \text{Mod}(\mathcal{L})$ lies in the Borel hierarchy then $X$ is approximated by simpler sets. Unfortunately the simpler sets will not be isomorphism-closed. We thus look for a stronger statement which applies even to sets $X$ which are not isomorphism-closed.

**Definition**

If $X \subset \text{Mod}(\mathcal{L})$ and $\vec{a} \in (\mathbb{N})^k$ then the *Vaught transform* $X^{*\vec{a}}$ is the set \{ $M \mid \forall^* g \in S_\infty (\vec{a} \subset g \implies gM \in X)$ \}.

**Theorem**

*If $X \subset \text{Mod}(\mathcal{L})$ is Borel and $k \in \mathbb{N}$ then there is a formula $\phi$ of $\mathcal{L}_{\omega_1\omega}$ with $k$ free variables such that $M \in X^{*\vec{a}} \iff \phi^M(\vec{a})$.*

López-Escobar’s theorem follows from the special case when $k = 0$. 
Outline of Vaught's proof of the stronger statement

Theorem
If $X \subset \text{Mod}(L)$ is Borel and $k \in \mathbb{N}$ then there is a formula $\phi$ of $L_{\omega_1 \omega}$ with $k$ free variables such that $M \in X^* \bar{a} \iff \phi^M(\bar{a})$.

Proof outline.
One shows that the class $\mathcal{K}$ of sets $X$ satisfying the conclusion satisfies:

- $\mathcal{K}$ contains the subbasic open sets
- $\mathcal{K}$ is closed under complementation
- $\mathcal{K}$ is closed under countable intersections

These are shown by calculations using the elementary properties of the Vaught transforms.
Metric structures

We seek analogs of the classical results above within the beautiful theory of metric structures and continuous logic.

Definition
A relational metric structure consists of:
- A complete metric space \((M, d)\) of diameter 1
- Relations \(R_i : M^n \to [0, 1]\), each uniformly continuous
  (the modulus of continuity is specified in the language)

Motivation
The \(R_i\) are grey sets. If \(R_i(\bar{a}) = 0\) then \(\bar{a}\) is surely in \(R_i\), and if \(R_i(\bar{a}) > 0\) then its value measures the failure.
The space of separable metric structures

We will confine ourselves to metric structures whose underlying metric space is the Urysohn sphere $\mathbb{U}$, that is, the universal ultrahomogeneous separable metric space of diameter 1.

**Definition**
If $\mathcal{L}$ is a countable metric language with symbols $R_i$ of arity $n_i$ and modulus $\Delta_i$, then we define the space of separable $\mathcal{L}$-structures

$$\text{MMod}(\mathcal{L}) = \prod \text{Unif}_{\Delta_i}(\mathbb{U}^{n_i}, [0, 1]).$$

Here $\text{Unif}_{\Delta}(X, Y)$ denotes the space of $\Delta$-uniformly continuous functions from $X$ to $Y$ with the topology of pointwise convergence.

**Remark**
The Polish group $\text{Iso}(\mathbb{U})$ of isometric bijections of $\mathbb{U}$ acts naturally on $\text{MMod}(\mathcal{L})$, and its orbits are the isomorphism classes.
Universality of $\text{MMod}(\mathcal{L})$

As in the classical case, the space $\text{MMod}(\mathcal{L})$ is a universal $\text{Iso}(\mathcal{U})$-space. In fact something stronger is true:

**Proposition (Ivanov–Majcher-Iwanow)**

Let $\mathcal{L}$ be a metric language with 1-Lipschitz symbols of arbitrarily large arity. Then for any Polish group $G$ and Borel action $G \curvearrowright X$, there exists an equivariant embedding:

$$\alpha: G \leq \text{Iso}(\mathcal{U}), \quad i: X \hookrightarrow \text{MMod}(\mathcal{L})$$

such that $i$ maps distinct orbits to distinct orbits.

**Proof idea.**

One uses the standard result of Uspenskij that $\text{Iso}(\mathcal{U})$ is a universal Polish group, together with the methods of the Becker–Kechris result for the classical logic action.
Continuous logic

Definition
Formulas of continuous logic (the logic for metric structures) are built from

- **atomic formulas**: the usual $R_i(x_1, \ldots, x_n)$, and the relation $d(x_1, x_2)$, which replaces equality
- **connectives**: for formulas $\phi_1, \ldots \phi_n$ we may form continuous combinations $f(\phi_1, \ldots, \phi_n)$
- **quantifiers**: for a formula $\phi$ we may form $\inf_x \phi(x, \ldots)$ and $\sup_x \phi(x, \ldots)$

The infinitary continuous logic $L_{\omega_1\omega}$ additionally allows

- **countable conjunctions and disjunctions**: if $\phi_n$ are formulas we may form $\inf_n \phi_n$ and $\sup_n \phi_n$
  (Note: We require the moduli of uniform continuity of the $\phi_n$ be uniformly bounded.)
Main result: López-Escobar for metric structures

It is easy to see by induction on complexity that if $\phi$ is a sentence of $\mathcal{L}_{\omega_1\omega}$ then the evaluation map $M \mapsto \phi^M$ is Borel. Ivanov–Majcher-Iwanow asked whether the converse holds.

**Theorem (López-Escobar for MMod)**

*If $X : \text{MMod}(\mathcal{L}) \to [0, 1]$ is a Borel and isomorphism-invariant grey set, then there exists a sentence $\phi$ of $\mathcal{L}_{\omega_1\omega}$ such that for all $M \in \text{MMod}(\mathcal{L})$ we have $X(M) = \phi^M$.*

**Remark**

If $X$ is $0,1$-valued we can additionally ensure that $\phi$ is $0,1$-valued. It follows that if $X$ is a genuine Borel and invariant subset of $\text{MMod}(\mathcal{L})$ then $X$ is axiomatized by a sentence of $\mathcal{L}_{\omega_1\omega}$. 

A model-theoretic characterization of TVC

Corollary

The TVC is equivalent to the variant of Vaught’s conjecture for classes axiomatized by a sentence of continuous $\mathcal{L}_{\omega_1\omega}$.

Proof.

$(\Rightarrow)$ This holds because $\text{MMod}(\phi)$ is always Borel.

$(\Leftarrow)$ Let $X$ be a standard Borel $G$-space.

- There exists a language $\mathcal{L}$ and an equivariant embedding $i: X \hookrightarrow \text{MMod}(\mathcal{L})$.
- By the main result, there exists a sentence $\phi$ of $\mathcal{L}_{\omega_1\omega}$ such that $i(X) = \text{MMod}(\phi)$.
- By the VC for $\mathcal{L}_{\omega_1\omega}$, the image $i(X)$ has countably or perfectly many classes.
- Hence $X$ has countably many or perfectly many orbits.
Idea of the proof of the main result

As in Vaught’s proof of López-Escobar’s theorem, we wish to find a stronger version for grey sets which are not necessarily invariant.

**Theorem**
If $X \subseteq \text{Mod}(\mathcal{L})$ is Borel and $k \in \mathbb{N}$ then there is a formula $\phi$ of $\mathcal{L}_{\omega_1\omega}$ with $k$ free variables such that $M \in X^{*\bar{a}} \iff \phi^M(\bar{a})$.  

**Theorem**
If $X: \text{MMod}(\mathcal{L}) \to [0, 1]$ is a Borel grey set and $k \in \mathbb{N}$, then there is a formula $\phi$ of continuous $\mathcal{L}_{\omega_1\omega}$ with $k$ free variables such that for all $\bar{a} \in (\mathbb{U})^k$ we have $X^{*\bar{a}}(M) = \phi^M(\bar{a})$.  

The TVC and metric structures Samuel Coskey (Boise State University)
Definitions needed for the stronger statement

Theorem

If \( X : \text{MMod}(\mathcal{L}) \rightarrow [0, 1] \) is a Borel grey set and \( k \in \mathbb{N} \), then there is a formula \( \phi \) of continuous \( \mathcal{L}_{\omega_1\omega} \) with \( k \) free variables such that for all \( \bar{a} \in (\mathbb{U})^k \) we have \( X^{\bar{a}}(M) = \phi^M(\bar{a}) \).

Definition (Ivanov–Majcher-Iwanow)

- Let \( d_i \) enumerate a fixed dense subset of \( \mathbb{U} \).
- basic grey sets of \( \text{Iso}(\mathbb{U}) \): for \( \bar{a} \in (\mathbb{U})^k \) let 
  \[ [\bar{a}](g) = d(g^{-1}(d_1 \ldots d_k), \bar{a}). \]
- grey category quantifiers: For \( X \) a grey subset of \( A \times B \) let 
  \( (\sup_{b \in B}^* X)(a) > r \iff \exists^* b \in B X(a, b) > r. \)
- grey Vaught transforms: For \( X \) a grey subset of \( \text{MMod}(\mathcal{L}) \) let 
  \( X^{[\bar{a}]}(M) = \sup_{g \in G}^* (X(gM) - [\bar{a}](g)) \).
Outline of the proof of the stronger statement

**Theorem**

If $X: \text{MMod}(\mathcal{L}) \to [0, 1]$ is a Borel grey set and $k \in \mathbb{N}$, then there is a formula $\phi$ of continuous $\mathcal{L}_{\omega_1 \omega}$ with $k$ free variables such that for all $\bar{a} \in (\mathbb{U})^k$ we have $X^*\bar{a}(M) = \phi^M(\bar{a})$.

**Proof outline.**

One again shows that the class $\mathcal{K}$ of grey sets $X$ satisfying the conclusion satisfies:

- $\mathcal{K}$ contains sufficiently many “basic” grey sets
- $\mathcal{K}$ is closed under negation $r - X$
- $\mathcal{K}$ is closed under linear combinations
- $\mathcal{K}$ is closed under $\sup_n$ and $\inf_n$

To conclude that $\mathcal{K}$ must contain all Borel functions, one argues as in the Lebesgue–Hausdorff theorem for Baire class functions.