Vaught’s Conjecture and Descriptive Set Theory

Su Gao

Department of Mathematics
University of North Texas

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Day 1

1. Vaught’s Conjecture

2. The Scott Analysis

3. Martin’s Conjecture

4. The Infinitary Vaught’s Conjecture

*Can it be proved, without the use of the continuum hypothesis, that there exists a complete theory having exactly $\aleph_1$ non-isomorphic denumerable models?*
Vaught’s Conjecture (Vaught, 1961)

Let $T$ be a complete first-order theory in a countable language. Then $T$ has either at most countably many or $2^\aleph_0$ (perfectly) many nonisomorphic countable models.
Fix a countable language $L$. Define

$$(M, \bar{a}) \equiv_0 (N, \bar{b}) : qftp_M(\bar{a}) = qftp_N(\bar{b})$$

$$(M, \bar{a}) \equiv_{\alpha+1} (N, \bar{b}) : \forall c \in M \exists d \in N (M, \bar{a} \upharpoonright c) \equiv_\alpha (N, \bar{b} \upharpoonright d)$$

and vice versa

$$(M, \bar{a}) \equiv_{\lambda} (N, \bar{b}) (\lambda \text{ limit}) : \forall \alpha < \lambda (M, \bar{a}) \equiv_\alpha (N, \bar{b})$$

$$(M, \bar{a}) \equiv_{\infty} (N, \bar{b}) : \forall \alpha (M, \bar{a}) \equiv_\alpha (M, \bar{b})$$
The back-and-forth equivalence relation $\equiv_\alpha$, $\alpha < \omega_1$, can be captured by $L_{\omega_1 \omega}$ formulas:

$$\varphi_{\alpha+1}^M(a, x) := \varphi_{\alpha}^M(a, x) \land \forall y \bigwedge_{c \in M} \exists y \varphi_{\alpha}^M(a, c, x, y) \land \forall y \bigvee_{c \in M} \varphi_{\alpha}^M(a, c, x, y)$$

$\varphi_{\alpha < \lambda}^M(a, x) := \bigwedge_{\alpha < \lambda} \varphi_{\alpha}^M(a, x)$

**Fact** $(M, a) \equiv_\alpha (N, b)$ iff $\varphi_{\alpha}^M(a) = \varphi_{\alpha}^N(b)$

Each $\varphi_{\alpha}^M(a)$ has quantifier rank exactly $2 \cdot \alpha$
Given an $L$-structure $M$.

For each tuple $\bar{a} \in M$, define

$$\rho_M(\bar{a}) = \inf\{\alpha : \forall \bar{b} \in M \left[ (M, \bar{a}) \equiv_\alpha (M, \bar{b}) \Rightarrow (M, \bar{a}) \equiv_\infty (M, \bar{b}) \right] \}$$

The Scott rank of $M$ is defined as

$$sr(M) = \sup\{\rho_M(\bar{a}) + 1 : \bar{a} \in M\}$$
Let \( M \) be countable. Then \( \text{sr}(M) < \omega_1 \).

The canonical Scott sentence of \( M \) is

\[
\varphi_M := \varphi_\alpha^M,\emptyset \land \bigwedge_{\bar{a} \in M} \forall \bar{x} \left[ \varphi_\alpha^M, \bar{a}(\bar{x}) \rightarrow \varphi_{\alpha+1}^M, \bar{a}(\bar{x}) \right]
\]

where \( \alpha = \text{sr}(M) \).

**Theorem (Scott)**

If \( N \) is countable, then \( N \models \varphi_M \) iff \( N \cong M \).

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The Scott Rank


**Theorem (Nadel, 1974)**
For any countable $M$, $sr(M) \leq \omega_1^M + 1$. 
Morley’s Theorem


**Theorem** (Morley, 1970)
Let \( T \) be a complete first-order theory in a countable language. Then \( T \) has either at most \( \aleph_1 \) many or \( 2^{\aleph_0} \) (perfectly) many nonisomorphic countable models.

He introduced the notion of scattered theories, i.e. those with \( S_n(T, F) \) countable for all countable fragments \( F \) and \( n < \omega \), and showed that a scattered theory has at most \( \aleph_1 \) many nonisomorphic countable models.
Martin’s Conjecture

Let $T$ be a complete first-order theory in a countable language. Define $L_1(T)$ to be the smallest fragment of $L_{\omega_1\omega}$ containing

$$L_{\omega \omega} \cup \left\{ \bigwedge_{\phi \in p} \phi(\bar{x}) : p \in S_n(T), n < \omega \right\}.$$

Martin asked if the following strengthening of Vaught’s Conjecture was known to be false:

*If $T$ is such that $S_n(T)$ is countable for all $n < \omega$, and if $T'$ is any completion of $T$ with respect to $L_1(T)$, then either $T'$ is $\aleph_0$-categorical or $T'$ has $2^{\aleph_0}$ many nonisomorphic countable models.*
Morley gave a counterexample (immediately):

The theory of an equivalence relation with infinitely many equivalence classes, each containing a model of an algebraically closed field of characteristic 0.
Martin’s Conjecture (Martin–Morley, early 1970s)

Let $T$ be a complete first-order theory in a countable language. If $T$ has fewer than $2^\aleph_0$ many nonisomorphic countable models, then for each countable model $M$ of $T$, letting $T_1(M)$ be the complete theory of $M$ in the language $L_1(T)$, $T_1(M)$ is $\aleph_0$-categorical.

This implies that, if $T$ has fewer than $2^\aleph_0$ nonisomorphic countable models, then the Scott ranks of all countable models of $T$ are $\leq \omega + \omega$. 

**Theorem** (Shelah–Harrington–Makkai, 1984)
Vaught’s Conjecture holds for $\omega$-stable theories.

In fact, their proof gives that, if $T$ has fewer than $2^{\aleph_0}$ nonisomorphic countable models, then the Scott ranks of all countable models of $T$ are $<\omega^2$.

**Theorem** (Bouscaren, 1984)
Martin’s Conjecture holds for $\omega$-stable theories.
The Strong Martin Conjecture (Wagner, 1982)

In addition to Martin’s Conjecture, if $T$ has $2^\aleph_0$ non-isomorphic countable models, then there are $2^\aleph_0$ distinct completions of $T$ in $L_1(T)$.

In 2001, I came across a counterexample:

The theory of infinite atomic Boolean algebras.
**Theorem (Tarski)**
There are only countably many complete consistent first-order theories of Boolean algebras.

**Fact** Martin’s Conjecture holds for Boolean algebras. In fact, every complete consistent first-order theory of Boolean algebras has either exactly one or $2^{\aleph_0}$ countable models.

**Theorem (G.)**
Let $A$ and $B$ be infinite atomic Boolean algebras. Then $A \equiv_{\omega+\omega} B$ iff $A/F(A) \equiv B/F(B)$, where $F(M)$ is the ideal generated by the atoms of $M$. In particular, there are only countably many complete extensions of the theory of infinite atomic Boolean algebras in $L_{\omega+\omega,\omega}$. 
The $L_{\omega_1 \omega}$ Vaught’s Conjecture

$L_{\omega_1 \omega}$ Vaught’s Conjecture

Let $L$ be a countable language. Any $L_{\omega_1 \omega}$ sentence $\varphi$ has either at most countably many or $2^{\aleph_0}$ (perfectly) many nonisomorphic countable models.

Morley’s theorem generalizes immediately to the $L_{\omega_1 \omega}$ setting.
Let $L$ be a countable language and $\varphi$ be a sentence in $L_{\omega_1 \omega}$. Let $F$ be the smallest fragment containing $\{\varphi\} \cup L_{\omega \omega}$ and $L_F$ be the smallest fragment containing

$$F \cup \left\{ \bigwedge_{\phi \in p} \phi : p \in S_n(\varphi, F), n < \omega \right\}.$$

For each countable model $M$ of $\varphi$, let $T_F(M)$ be the complete theory of $M$ in $L_F$. If $\varphi$ has fewer than $2^{\aleph_0}$ nonisomorphic countable models, then $T_F(M)$ is $\aleph_0$-categorical for all countable models $M$ of $\varphi$. 
The $L_{\omega_1 \omega}$ Martin’s Conjecture implies that, if $\varphi$ has fewer than $2^{\aleph_0}$ nonisomorphic countable models, then the Scott ranks of all countable models of $\varphi$ are $\leq qr(\varphi) + \omega$. 

**Theorem (Steel, 1978)**

The $L_{\omega_1\omega}$ Vaught’s Conjecture holds for “trees,” i.e., partially ordered sets so that the set of predecessors of any element is linearly ordered.

This generalized earlier results:

- **Marcus, A. Miller**: first-order theories of mono-unary algebras
- **Rubin**: first-order theories of linear orders
Theorem (Wagner, 1982)
The Strong Martin Conjecture holds for

- first-order theories of mono-unary algebras;
- first-order theories of linear orders.

Theorem (Wagner, 1982)
If $T$ is a first-order theory of trees and $T$ has fewer than $2^{\aleph_0}$ nonisomorphic countable models, then the Scott rank of any model $M$ of $T$ is $< \omega^2$. 

**Theorem (Sacks)**
For any sentence $\varphi$ in $L_{\omega_1\omega}$, if $\text{sr}(M) < \omega_1^M$ for all countable models $M$ of $\varphi$, then there is $\alpha < \omega_1$ such that $\text{sr}(M) < \alpha$ for all countable models $M$ of $\varphi$. In particular, Vaught’s Conjecture for $\varphi$ holds.

**Theorem (Makkai, 1981)**
For any scattered sentence $\varphi$ in $L_{\omega_1\omega}$, if for every countable model $M$ of $\varphi$, the sentence

$$\bigwedge\{\phi \in L_{\omega_1\omega} \cap \text{HYP}(M) : M \models \phi\}$$

is a Scott sentence of $M$, then there is $\alpha < \omega_1$ such that $\text{sr}(M) < \alpha$ for all countable models of $M$ of $\varphi$. 

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**Theorem** *(Sacks, ≤1970)*

For any scattered sentence \( \varphi \) in \( L_{\omega_1\omega} \), if

- \( \text{sr}(M) \leq \omega_1^M \) for all countable models \( M \) of \( \varphi \), OR
- for each \( \alpha < \omega_1 \) there is at most one countable model of \( \varphi \) of Scott rank (exactly) \( \alpha \),

then \( \varphi \) has only countably many countable models.

Under either of the above assumptions, the countable models of \( \varphi \) all have Scott ranks \( \leq \sigma_2^\varphi + 1 \), where \( \sigma_2^\varphi \) is the least \( \Sigma_2 \) admissible ordinal relative to \( \varphi \).
The following problems seem still of interest:

- The $L_{\omega_1 \omega}$ Vaught’s Conjecture for Boolean algebras

- Any positive instance of the $L_{\omega_1 \omega}$ Martin’s Conjecture (possibly with small bounds replacing $\omega$)

- Martin’s Conjecture for trees
Day 2

1. The Topological Vaught’s Conjecture

2. Dichotomies for Equivalence Relations

3. Results on the Topological Vaught’s Conjecture

4. Borel Reducibility and Benchmark Equivalence Relations

5. Borel Isomorphism Relations
Topological methods had been introduced into the subject early on, by Vaught and his students:


Miller proposed the following problem in 1980:
Suppose a Polish group $G$ acts continuously (or Borel measurably) on a Polish space $X$. Let $B \subseteq X$ be an invariant Borel set containing uncountably many orbits.

**Conjecture** There is a perfect set of $G$-inequivalent elements of $B$.

This is known to be so if $G$ is locally compact. The following special case is of particular interest: $G = \mathbb{N}!$ (the permutation of $\mathbb{N}$), $X = 2^{\mathbb{N} \times \mathbb{N}}$, with the canonical action $g\alpha(n, m) = \alpha(gn, gm)$ for $\alpha \in 2^{\mathbb{N} \times \mathbb{N}}$. 
The Topological Vaught’s Conjecture (phrase coined by Sami)

Let $G$ be a Polish group with a Borel action on a Polish space $X$. Let $B \subseteq X$ be an invariant Borel subset of $X$. Then either $B$ contains at most countably many orbits or $B$ contains perfectly many orbits.

He introduced the **Vaught transforms**: for $A \subseteq X$ and $V \subseteq G$,

\[ A^{\Delta V} := \{ x \in X : \exists^* g \in V \ g \cdot x \in A \} \]

\[ A^{*V} := \{ x \in X : \forall^* g \in V \ g \cdot x \in A \} \]
**Theorem** (Becker–Kechris, early 1990s)
The Topological Vaught’s Conjecture for $S_\infty$ is equivalent to the $L_{\omega_1\omega}$ Vaught’s Conjecture.

**Theorem** (Becker–Kechris, early 1990s)
The Topological Vaught’s Conjecture is equivalent to the statement

> For any continuous action of a Polish group on a Polish space, either there are at most countably many orbits or there are perfectly many orbits.

Several key ingredients of the Becker–Kechris theory:

▶ Change-of-topology technique
▶ Universal actions of Polish groups
▶ Coding of arbitrary $S_\infty$ actions by logic actions
The Silver Dichotomy (Silver)
Let $E$ be a coanalytic equivalence relation on a Polish space $X$. Then $E$ has either at most countably many classes or there are perfectly many $E$-classes.

**Theorem (Burgess)**
Let $E$ be an analytic equivalence relation on a Polish space $X$. Then $E$ has either at most $\aleph_1$ many classes or there are perfectly many $E$-classes.
Burgess’s theorem is sharp: there are analytic equivalence relations with exactly $\aleph_1$ many equivalence classes but not perfectly many equivalence classes.

Consider the equivalence relation on $\omega^\omega$ defined by

$$\omega^x_1 = \omega^y_1$$

Why is it not a counterexample to the Topological Vaught’s Conjecture?

Marker showed that it is not an orbit equivalence relation. He consider the set

$$X_{low} = \{x \in \omega^\omega : \omega^x_1 = \omega^y_1^{CK}\}$$

and observed that it is nonmeager but not $G_\delta$. 
Results on the Topological Vaught’s Conjecture

The Scott analysis implies that, in the case of the isomorphism relation, we have for every $x$,

$$[x] \in \Pi^0_{\omega_1^x+2}$$

Sami generalized this to arbitrary Polish group actions.

**Theorem (Sami)**

Let $G \acts X$ be a $\Delta^1_1$ action of a (recursively presented) Polish group on a (recursively presented) Polish space. Let $A \subseteq X$ be a $\Sigma^1_1$ invariant set. Suppose for all $x \in A$, $[x] \in \Pi^0_{\omega_1^x}$. Then either $A$ has at most countably many orbits or $A$ contains perfectly many orbits.

This generalizes a result of Makkai for isomorphism relations. Makkai has noted that the condition does not imply the relation is Borel, hence Silver’s theorem does not apply.
Theorem (Steel)
Assume $\Sigma^1_1$ determinacy. Suppose $E$ is a $\Sigma^1_1$ equivalence relation and for every $x$, $[x] \in \Sigma^0_{\omega_1^x + 2}$. Then either $E$ has at most countably many classes or $E$ contains perfectly many classes.
The Topological Vaught’s Conjecture for Restricted Classes of Groups

**Folklore** The Topological Vaught’s Conjecture holds for locally compact groups.

**Theorem (Sami)**
The Topological Vaught’s Conjecture holds for abelian groups.

**Theorem (Hjorth–Solecki)**
The Topological Vaught’s Conjecture holds for nilpotent groups and TSI groups.

**Theorem (Becker)**
The Topological Vaught’s Conjecture holds for CLI groups.
In each of these cases the result was shortly or immediately after extended to analytic sets.

**Theorem (Hjorth)**
Let $G$ be a Polish group. The following are equivalent:

(i) There is a Borel action of $G$ on a Polish group $X$ and a $\Sigma^1_1$ $G$-invariant subset $A \subseteq X$ containing exactly $\aleph_1$ many $G$-orbits.

(ii) There is a closed subgroup of $G$ that has $S_\infty$ as a continuous homomorphic image.
Borel Reducibility

For equivalence relations $E, F$ on Polish spaces $X, Y$, resp., a **Borel reduction** from $E$ to $F$ is a Borel measurable function $f : X \to Y$ such that for all $x, y \in X$,

$$x Ey \iff f(x) F f(y).$$

If there exists a Borel reduction from $E$ to $F$, we write $E \leq_B F$.

Also define $E <_B F$, $E \sim_B F$, etc.
id(ω): identity (equality) relation on ω

In the context of the Topological Vaught’s Conjecture, i.e., a Polish group $G$ with a Borel action on a Polish space $X$, giving rise to the orbit equivalence relation $E$, the following are equivalent:

- $E$ has at most countably many orbits;
- $E \leq_B \text{id}(ω)$. 
id(2^\omega): identity (equality) on 2^\omega

For any equivalence relation \( E \) on a Polish space \( X \), the following are equivalent:

- There are perfectly many \( E \)-orbits, i.e., there is a perfect set of \( E \)-inequivalent elements;
- \( \text{id}(2^\omega) \sqsubseteq_c E \);
- \( \text{id}(2^\omega) \leq_B E \).
Topological Vaught’s Conjecture restated:

Let $G$ be a Polish group with a Borel action on a Polish space $X$. Let $B \subseteq X$ be an invariant Borel subset of $X$ and $E$ be the orbit equivalence relation on $B$. Then either $E \leq_B \text{id}(\omega)$ or else $\text{id}(2^\omega) \leq_B E$. 
The Friedman–Stanley tower:

\[=^+: \text{“equality of countable sets of reals”: for } x, y \in 2^{\omega \times \omega},\]

\[x =^+ y \iff \forall n \exists m \forall k x(n, k) = y(m, k) \land \text{v.v.}\]

Similarly, can define \(=^{\alpha+}\) for all \(\alpha < \omega_1\).

\[\text{id}(\omega) <_B \text{id}(2^{\omega}) <_B =^+ <^{2+} <_B \cdots\]
Let $L$ be a countable relational language and $\varphi$ be a sentence in $L_{\omega_1 \omega}$.

$\text{Mod}(\varphi)$: the set of all countable models of $\varphi$ with universe $\omega$

$\equiv_\varphi$: $\equiv$ on $\text{Mod}(\varphi)$

- $\equiv_\varphi$ is an $S_\infty$-orbit equivalence relation;
- $\equiv_\varphi$ is analytic, not necessarily Borel;
- Each $\equiv_\varphi$ orbit is Borel.
Isomorphism Relations

For each $\alpha < \omega_1$,

- the back-and-forth equivalence relation $\equiv_\alpha$ is a Borel equivalence relation;
- Scott formulas are essentially hereditarily countable sets; thus $\equiv_\alpha \leq B^{\alpha+}$
Theorem The following are equivalent:

- $\cong_\varphi$ is Borel;
- There is $\alpha < \omega_1$ such that $\cong_\varphi$ coincides with $\equiv_\alpha$;
- There is $\alpha < \omega_1$ such that $\cong_\varphi \leq B \equiv_\alpha$;
- There is $\alpha < \omega_1$ such that $\cong_\varphi \leq B = \alpha^+$;
- There is $\alpha < \omega_1$ such that $\text{sr}(M) < \alpha$ for all countable models $M$ of $\varphi$;
- There is $\beta < \omega_1$ such that for all countable models $M$ of $\varphi$, if $\text{sr}(M) > \beta$ then $\text{sr}(M) < \omega_1^M$.
- For all countable models $M$ of $\varphi$ of sufficiently high Scott rank, there is a Scott sentence of $M$ in $\text{HYP}(M)$. 
Day 3

1. More Benchmark Equivalence Relations

2. Results on the Glimm–Effros Dichotomy

3. Dichotomies for Isomorphism Relations

4. Counterexamples to the Topological Vaught’s Conjecture
$E_0$: for $x, y \in 2^\omega$, 

$$xE_0y \iff \exists m \forall n \geq m \ x(n) = y(n)$$

$id(2^\omega) <_B E_0 <_B =^+$
The Glimm–Effros Dichotomy

(Harrington–Kechris–Louveau)

Let $E$ be a Borel equivalence relation on a Polish space $X$. Then either $E \leq_B \text{id}(2^\omega)$ or $E_0 \leq_B E$.

The Glimm–Effros Dichotomy is known to be false for isomorphism relations in general. Consider countable torsion abelian groups. They are classifiable by Ulm invariants. $E_0$ does not embed into their isomorphism.
**Theorem (Hjorth–Solecki)**
The Glimm–Effros Dichotomy holds for nilpotent groups and TSI groups. In fact, either $E_0$ embeds continuously or else every orbit is $G_δ$.

**Theorem (Becker)**
The Glimm–Effros Dichotomy holds for CLI groups. In fact, either $E_0$ embeds continuously or else every orbit is $Π^0_ω$. 

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Glimm–Effros Dichotomy for Isomorphism Relations

Theorem (G.)
The Glimm–Effros Dichotomy holds for linear orders, i.e., for any $L_{\omega_1 \omega}$ sentence $\varphi$ all of whose models are linear orders, either $\equiv_\varphi \leq_B \text{id}(2^\omega)$ or $E_0 \leq_B \equiv_\varphi$.

Theorem (G.)
The Glimm–Effros Dichotomy holds for mono-unary algebras.
Theorem (Hjorth–Kechris, Becker)
Let $G \ltimes X$ be a Borel action of a Polish group on a Polish space. Let $E$ be the orbit equivalence relation. Then either $E_0 \sqsubseteq_c E$ or else there is a $C$-measurable reduction function from $E$ into the space of countable sequences of countable ordinals ($E \leq_c 2^{<\omega_1}$).
Let $G \curvearrowright X$ be a Borel action of a Polish group on a Polish space. Let $E$ be the orbit equivalence relation. Then either the Glimm–Effros Dichotomy holds for $G$ or else there is a closed subgroup of $G$ with $S_\infty$ as a continuous homomorphic image.
**Theorem** *(Hjorth–Kechris)*

Assume $\Sigma^1_1$ determinacy. For any $\Sigma^1_1$ equivalence relation $E$ on a Polish space, either $E_0 \subseteq^c E$ or else there is a $\Delta^1_2$ reduction function from $E$ into $2^{<\omega_1}$. 
**Theorem (Hjorth)**
Assume $\Sigma^1_1$-determinacy. For any $\Pi^1_1$ equivalence relation $E$ on a Polish space, either $E \leq B E$ or else there is a $\Delta^1_3$ reduction function from $E$ into $2^{<\omega_1}$. 
An equivalence relation $E$ is **Borel-complete** if both

- $E \sim_B$ some $S_\infty$-orbit equivalence relation, and
- any $S_\infty$-orbit equivalence relation $\leq_B E$.

**Theorem (Friedman–Stanley, 1989)**
The isomorphism relations for the following classes of countable structures are Borel-complete:

- mono-ary algebras
- linear orders
- (nilpotent) groups (of rank 2)
- fields
Denote any Borel-complete equivalence relation by $E_{S_\infty}$.

\[\text{id}(\omega) <_B \text{id}(2^\omega) <_B E_0 <_B -^+ <_B \cdots <_B =^\alpha + <_B \cdots <_B E_{S_\infty}\]

All but the last one are Borel equivalence relations.
Theorem (Schirmann)
Let $T$ be a complete first-order theory of linear orders. Then either $T$ is $\aleph_0$-categorical or else $\cong_T$ is Borel-complete.

Theorem (Camerlo–G.)
Let $T$ be a complete first-order theory of Boolean algebras. Then either $T$ is $\aleph_0$-categorical or else $\cong_T$ is Borel-complete.
A Friedman–Stanley Conjecture?

Is there an $L_{\omega_1 \omega}$ sentence $\varphi$ so that $\models \alpha^+ \leq_B \equiv_{\varphi}$ for all $\alpha < \omega_1$ and yet $\equiv_{\varphi}$ is not Borel-complete?

**Theorem (G.)**

Let $\varphi$ be an $L_{\omega_1 \omega}$ sentence in the language of mono-unary algebras. Then either $\equiv_{\varphi}$ is Borel or $\equiv_{\varphi}$ is Borel-complete.
**Theorem** (Becker)
Let $G \curvearrowright X$ and $A \subseteq X$ be a counterexample to the Topological Vaught’s Conjecture. Then there is a Borel $B \subseteq A$ that is a minimal counterexample.
Becker proved:

If \((G, X, A)\) is a counterexample to TVC, then

- there is a club of \(\alpha \in \omega_1\) such that \(A\) contains an orbit

\[
[x] \in \prod_{\alpha+1}^0 \setminus \bigcup_{\beta<\alpha} \prod_{\beta}^0
\]

- for all sufficiently large limit \(\alpha < \omega_1\), \(A\) contains an orbit as above.

If \((G, X, A)\) is a minimal counterexample to TVC, then

- there is a club of \(\alpha \in \omega_1\) such that \(A\) contains exactly one orbit

\[
[x] \in \prod_{\alpha+1}^0 \setminus \bigcup_{\beta<\alpha} \prod_{\beta}^0
\]
Becker proved:

If \((G, X, A)\) is a counterexample to TVC, then there is \(a \in 2^\omega\) such that for all \(v \geq_T a\), there is a club of \(\alpha \in \omega_1\) such that \(A\) contains an orbit \([x]\) with the properties

(i) \(\omega_1^{[x]} = \alpha\), and

(ii) for all \(u \geq_T v\), if \(\omega_1^u = \alpha\) then there is \(y \in [x]\) with \(\omega_1^{u,y} = \alpha\)

(Assume \(\Delta^1_3\) determinacy) If \((G, X, A)\) is a minimal counterexample to TVC, then there is a club of \(\alpha \in \omega_1\) such that \(A\) contains exactly one orbit with the above properties.
Theorem (Hjorth)

Let $\varphi$ be a counterexample to the $L_{\omega_1\omega}$ Vaught’s Conjecture and let $M$ be any countable $L$-structure with $\text{Aut}(M)$ large (meaning there is a closed subgroup of $G$ with $S_\infty$ as a continuous homomorphic image). Then there is a counterexample $\sigma$ (in a language that is an expansion of $L$) such that $\sigma \rightarrow \varphi_M$ (i.e., every model of $\sigma$ is an expansion of $M$).