Research Statement

October 2, 2015

My research is in the area of mathematical logic, specifically computability theory and its interactions with set theory. Broadly speaking, I seek to understand the information content of mathematical theorems and structures. Here, “information content” is meant in the context of Turing computability, and follows the general two-part Main Theme:

(Computable structure theory) If $A$ is a structure, we measure the complexity of $A$ by asking: what objects can we construct, computably, when given an arbitrary isomorphic copy of $A$ as an oracle?

(Reverse mathematics) If $T$ is a theorem, we measure the complexity of $T$ by asking: what axioms are needed to prove $T$, beyond those which correspond to “computable” mathematics?

Both parts of the Main Theme are well-understood in the context of $\mathbb{N}$ — where we consider countable structures (and look at copies coded by subsets of $\mathbb{N}$), and theorems in the language of second-order arithmetic (that is, about natural numbers and sets of natural numbers, or about objects which can be represented as such). The main focus of my work is on the interaction between the Main Theme and set theory: what set theory can tell us about generalizing the Main Theme to broader contexts, and how the Main Theme is related to similar themes focused on set-theoretic notions of complexity.

My thesis will be composed of work from multiple articles ([1],[5],[7],[9],[11],[10]). Here I concentrate on three directions taken in [9],[7], and [5]; I will close by briefly summarizing the other articles. The first and third are from reverse mathematics and computable structure theory, respectively; the second is focused on the interaction between various notions of “constructibility” of a structure, and led to the third. All three involve generalizing concepts of classical computability theory to broader settings.

• [9]: Comparing the strength of determinacy for games played on $\mathbb{R}$, and related principles.

• [7] (With Knight and Montalbán): Analyzing when a structure which can be “forced” to exist must already exist.

• [5] (With Igusa and Knight): Defining and studying the algorithmic complexity of uncountable structures, and versions of the reals in particular.

Determinacy of games played on $\mathbb{R}$ [9]
Reverse mathematics studies the computational complexity of theorems of mathematics by examining what axioms are needed to prove them, over a base theory of axioms which are considered “computably true.” This theory is usually taken to be $RCA_0$, and reverse mathematics over $RCA_0$ has been extensively developed. However, it has severe limitations: in order to analyze a theorem’s strength over $RCA_0$, that theorem has to be expressible in the language of second-order arithmetic, or translatable into that language in a natural way. A base theory $RCA^0_\omega$ suitable for a broader class of theorems was introduced Kohlenbach [8]. $RCA^0_\omega$ is more technical and expressed very differently than $RCA_0$; to bridge the gap, I introduced a base theory $RCA^3_0$ in the language of third-order arithmetic, which is much closer to the original $RCA_0$, and showed that it is a conservative extension of $RCA^3_0$.

Using $RCA^3_0$, I studied the reverse mathematics of higher determinacy principles. A game on $X$ consists of a subset of $A \subseteq X^\omega$, viewed as the payoff set for an infinitely long game: players I and II alternate playing elements of $X$, and after $\omega$-many steps, I wins if the resulting sequence is in $A$. “Determinacy principles” — statements of the form “every game of the following type is determined” — play a central role in logic: they are intimately related with large cardinals, and imply strong “regularity” properties of sets corresponding to the games. Endow $X^\omega$ with the topology coming from the product topology, where each factor of $X$ carries the discrete topology. A clopen (open) game is one whose payoff set is clopen (open) in this topology. On $\mathbb{N}$, clopen and open determinacy are equivalent over $RCA_0$. However:

**Theorem 0.1** (S.). Over $RCA^3_0$, the statement “Every clopen game on $\mathbb{R}$ is determined” is strictly weaker than the statement “Every open game on $\mathbb{R}$ is determined.”

My proof used a generalization of tagged tree forcing. Shortly afterwards, Hachtman found a second proof using the fine structure of Goedel’s constructible universe $L$. There are many questions around this area which remain unanswered; for instance, it is unknown whether the two principles have the same second-order consequences.

**Structures in generic extensions** [7]

An important result of Solovay is that the only sets common to all forcing extensions are those which already exist. Roughly:

**Theorem 0.2** (Solovay). Fix a partial order $\mathbb{P}$. If $x$ is any set possibly outside the ground model, which is in every generic extension by $\mathbb{P}$, then $x$ is in the ground model already.

One may ask whether Solovay’s theorem holds for objects other than sets. Roughly:

**Question 1.** Fix a partial order $\mathbb{P}$. Suppose there is a structure $\mathcal{A}$, possibly outside $V$, such that $V[G]$ contains a structure isomorphic to $\mathcal{A}$ whenever $G$ is $\mathbb{P}$-generic; must $V$ contain an isomorphic copy of $\mathcal{A}$ already?

The case of countable structures, up to isomorphism, is particularly interesting because of possible connections with Vaught’s conjecture (see below). Examining countable structures in particular, Knight, Montalbán, and I showed that in general the answer is “no”: for some partial orders $\mathbb{P}$, there are “$\mathbb{P}$-generically presentable” structures which do not have copies in $V$. However, if $\mathbb{P}$ is not too wild, then the answer is yes, and in fact the question of whether $\mathbb{P}$-generically presentable structures necessarily have copies in $V$ is equivalent to the question of how much forcing with $\mathbb{P}$ distorts the cardinalities of infinite sets:
Theorem 0.3 (Knight, Montalbán, S. [?]). Fix a partial order $\mathbb{P}$. The following are equivalent:

There is a $\mathbb{P}$-generically presentable countable structure with no isomorphic copy in $V$.

There is a $\mathbb{P}$-generic filter $G$ such that the ordinal which (in $V$) is $\omega_2$ is countable in $V[G]$.

Shortly after, this was independently proved by Kaplan and Shelah [6].

We used this to give a new proof of a theorem of Harrington. Vaught’s conjecture is the statement that for any countable first-order theory $T$, if $T$ has uncountably many countable models, then $T$ has continuum-many countable models. Harrington showed that any counterexample to Vaught’s conjecture must have very complicated models of size $\aleph_1$:

Theorem 0.4. Any counterexample to Vaught’s conjecture has models of size $\aleph_1$ with Scott rank arbitrarily high below $\omega_2$.

A similar proof of Harrington’s theorem was given, around the same time, by Baldwin, Friedman, Koerwien, and Laskowski [2].

Uncountable computable structure theory [5]

There is a natural way to compare the information content of two countable structures:

Definition 1. For $\mathcal{A}, \mathcal{B}$ countable, say $\mathcal{A}$ is Muchnik reducible to $\mathcal{B}$ (and write $\mathcal{A} \leq_w \mathcal{B}$) if every copy of $\mathcal{B}$ with domain $\omega$ computes some copy of $\mathcal{A}$ with domain $\omega$.

In order to extend Muchnik reducibility to uncountable structures, we use set theory, specifically forcing. For $\kappa$ a cardinal, the partial order of finite sequences of elements of $\kappa$, ordered by extension, collapses $\kappa$ to $\omega$: if $G$ is generic for this partial order, then in $V[G]$, the ordinal $\kappa$ is countable. Taking $\kappa = \max\{|\mathcal{A}|, |\mathcal{B}|\}$ then allows us to define:

Definition 2 (S.). For $\mathcal{A}, \mathcal{B}$ structures of arbitrary cardinality, write $\mathcal{A} \leq^*_w \mathcal{B}$ if for some generic extension $V[G]$ in which $\mathcal{A}, \mathcal{B}$ are countable, we have $\mathcal{A} \leq_w \mathcal{B}$.

A standard metatheorem about forcing — Shoenfield’s absoluteness theorem — implies that “some” can be replaced with “all,” and that $\leq^*_w$ agrees with $\leq_w$ when both structures are countable, thus motivating this as a natural definition. This reducibility was introduced in [7]. Shortly afterwards, Igusa and Knight [4] began the project of looking at versions of the real numbers. They examined the structures $(\mathbb{R}; +, \times)$ and $(\mathbb{N}, \mathcal{P}(\mathbb{N}); 1, +, \in)$ — the field of real numbers, and Cantor space, both commonly referred to as “the reals” in logic — and showed that the former is strictly more complicated than the latter in terms of $\leq^*_w$. Following this, Igusa, Knight, and I looked at expansions of the field of reals. We showed that adding analytic functions does not increase the complexity of the structure:

Theorem 0.5. If $f : \mathbb{R} \to \mathbb{R}$ is analytic, then $(\mathbb{R}; +, \times, f) \leq^*_w (\mathbb{R}; +, \times)$

We also showed that various reducts of the field of reals have the same complexity. The proofs use model-theoretic techniques, specifically $\omega$-minimality and deep results of Macintyre and Wilkie, thus connecting several areas of logic at once. In addition, we showed that the complexity of another version of the reals — Baire space, $\mathbb{N}^\mathbb{N}$, viewed as a two-sorted structure — is again the same as $(\mathbb{R}; +, \times)$.
Further directions.
In [1], with Uri Andrews, Mingzhong Cai, and David Diamondstone, I examine a family of jump-like operators on the Turing degrees arising from ultrafilters; we classify exactly the possible behavior of the recursive sets under these operators, and this leads to a purely combinatorial proof of a classical theorem of reverse mathematics: that $WKL_0$ is strictly weaker than $ACA_0$. We also explore lowness notions related to ultrafilters, and the behavior of these operators on uncountable Turing ideals under set-theoretic hypotheses.

In [11], I explore versions of the axiom of choice from the perspective of higher reverse mathematics. This provides a finer method of comparison than equivalence over $ZF$, the usual approach to studying choice principles; in particular, I show that there is a model of $RCA_0^3$ in which the reals are well-ordered but real-indexed sets of reals need not have choice functions.

In [10], I explore the strength of Banach-Mazur determinacy principles. I show that Banach-Mazur determinacy principles for classes of Borel payoff sets correspond neatly to levels of hyperarithmetic comprehension, and Borel Banach-Mazur determinacy is equivalent to $ATR_0$. In addition, there is an upper bound on the complexity of reals which can be coded into their payoff sets: for $x$ a real, there is a determined Banach-Mazur games $G$ such that every winning strategy for $G$ computes $X$ if and only if $x$ is hyperarithmetic.

References


