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Quantifier Elimination for the Relative Frobenius

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Abstract. Let (K, v) be a complete discretely valued field of characteristic zero with an algebraically closed residue field of positive characteristic. Let $\sigma : K \to K$ be a continuous automorphism of K inducing a Frobenius automorphism on the residue field. We prove quantifier-elimination for (K, v, σ) in a language with angular component maps and in a language with predicates on leading terms. The proof passes through a generalization of the main Ax-Kochen-Eršov and quantifier-elimination results of [6] to a wider class of D-henselian fields of characteristic zero.

1 Introduction

A valued difference field is a valued field (K, v) given together with an automorphism $\sigma : K \to K$ preserving the valuation in the sense that $v(x) = v(\sigma(x))$ holds universally on $K^{\times}$. Examples of valued difference fields are legion though complete fields of positive residue characteristic given together with a relative Frobenius may be the mostly widely exploited.

Valued D-fields, a general framework for considering valued difference and differential fields, were introduced in [6] and quantifier elimination relative to the residue field and the value group for D-henselian fields with root-closed linearly differentially closed residue fields of characteristic zero was proved. However, quantifier elimination for valued difference fields eluded the methods of that paper for good reason: if the distinguished automorphism is nontrivial on the residue field, then the theory of (K, v, σ) *cannot* eliminate quantifiers, even relative to the residue field and the value group. The culprit is the same obstruction to quantifier elimination for henselian (pure) fields with residue fields not closed under roots: the existential quantifier defining the ℓ-th powers cannot be eliminated simply by expanding the language for the residue field and the value group.

While the formulas defining powers lie at the heart of the failure of quantifier elimination, it does not suffice to simply expand the language with power predicates.

However, the solution to the problem for D-henselian fields is no more difficult than the solution for henselian (pure) fields.

Ax and Kochen [1] approached this problem by expanding the language to include a section of the valuation. If (K, v) is a valued field and $\chi : vK \to K^\times$ is a section of the valuation, then we may write any element $x \in K^\times$ as $x = \frac{x}{\chi(v(x))}\chi(v(x))$. If K is henselian of residue characteristic zero, then x is an n-th power if and only if n divides $v(x)$ and the reduction of $\frac{x}{\chi(v(x))}$ is an n-th power in the residue field for any $n \in \mathbb{Z}_+$.

Expanding the language with a section of the valuation one substantially alters the class of definable sets. Such a section is never definable in the pure valued field language and some of the new definable sets are quite pathological in comparison to those definable in the valued field language. For instance, the image of the section is an infinite definable subset of the field which has no interior. While in a henselian field of characteristic zeor no set definable in the valued field language can have this property. For some valued fields, notably $\mathfrak{p}$-adic fields, it suffices to add power predicates [5], but in general the Basarab-Kuhlmann technology of additive multiplicative congruences (amc structures) is needed to obtain quantifier elimination in a definitional expansion of the valued field language [2].

While amc structures meet the condition of not expanding the class of definable sets, they may very well strike the reader as obscure (precise definitions are given in section 3). Angular component functions restore the transparency of the axioms in a language with a cross section while retaining the topological properties of the definable sets in the original language. An angular component function (of level zero for a pure valued field) is nothing more than a group homomorphism from the units of the field to the units of the residue field which is trivial on the 1-units and induces the identity on the residue field. If $\pi : \mathcal{O}_K \to \mathcal{O}_K/\mathfrak{m}_K$ is the residue map and $\chi : vK \to K^\times$ is a section of the valuation, then the associated angular component map is $x \mapsto \pi(\frac{x}{\chi(v(x))}) =: \alpha(x)$. For K a henselian field of residue characteristic zero, x is an n-th power if and only if $\alpha(x)$ is an n-th power and n divides $v(x)$. We use χ only to get α, the rest of the information supplied by χ is wasted.

In this work we modify the amc and angular component techniques to suit valued D- fields. We use angular component functions and amc structures of higher level to pass from mixed characteristic to pure characteristic zero.

The main theorem of this paper is an extension of the results of [6] to a complete axiomatization and quantifier elimination for a wider class of D-henselian fields. This class includes all D-henselian fields of characterization zero. Perhaps, the most important example of such a valued D-field is $(W_{p^\infty}(\mathbb{F}_p^{\mathrm{alg}})[\frac{1}{p}], \sigma_q)$, the field of fractions of the Witt vectors of $\mathbb{F}_p^{\mathrm{alg}}$ (also known as $\widehat{\mathbb{Q}_p^{\mathrm{unr}}}$, the completion of the maximal unramified extension of the p-adics) with the unique lifting of the q-power Frobenius. Independently from the current author, Luc Belair and Angus Macintyre obtained a version of the main theorem of this paper. Their work and its connections to the present paper will be reported in [4].

This paper is organized as follows. We recall the formalism of amc structures and angular components and adapt them to valued D-fields. We then set out the languages to be used and state precisely the theorems to be proved. We present a standard valuation coarsening argument to reduce to the study of valued D-fields of pure characteristic zero. The remainder of the paper consists of a detailed guide to modifying the arguments of [6] to the expanded languages.

I originally envisioned the results of the current paper as a part of my thesis [7], but realized through conversations with my thesis advisor, Ehud Hrushovski, that predicates beyond those used in [7] were necessary. A number of people contributed to my understanding of this problem. I thank especially Luc Belair (for suggesting that angular components may be the most natural framework for quantifier elimination), Lou van den Dries (for supplying a preprint of [8]), Ehud Hrushovski (for his advice during the work on [7]), Franz-Viktor Kuhlmann (for bringing amc structures to my attention), and Angus Macintyre (for discussing his work on the relative Frobenius). During the writing of this paper I was partially supported by an NSF MSPRF.

2 Notation

The notation used in this paper follows that of [6] with the exception that we revert to a more widely used notation for the value group of a valued field.

For us a ring is a unital ring. If R is a ring, then $R^\times := \{x \in R : (\exists y) xy = 1\}$ is the set of units in R considered as subgroup of the multiplicative monoid of R.

If K is a field with a valuation v, then we write vK for the value group of K and $\mathcal{O}_{K,v}$, or $\mathcal{O}_K$ is v is understood, for the ring of integers $\{x \in K : v(x) \geq 0\}$. The maximal ideal in $\mathcal{O}_{K,v}$ is $\mathfrak{m}_{K,v} = \mathfrak{m}_K := \{x \in K : v(x) > 0\}$. Sometimes we regard a valued field as a three-sorted structure (K, Γ_K, k_K) where $\Gamma_K = vK$ and $k_K = \mathcal{O}_{K,v}/\mathfrak{m}_{K,v}$ is the residue field. It will become necessary for us to liberate Γ_K and k_K so that we have only $vK \leq \Gamma_K$ and $\mathcal{O}_{K,v}/\mathfrak{m}_K \subseteq k_K$.

Recall that a D-ring is a commutative ring R given together with a fixed element $e \in R$ and a function $D : R \to R$ satisfying $D(1) = 0$, $D(x + y) = D(x) + D(y)$, and $D(xy) = D(x)y + xD(y) + eD(x)D(y)$ universally. On any D-ring there is an endomorphism $\sigma : R \to R$ defined by $x \mapsto eD(x) + x$. The set of D-constants, $R^D := \{x \in R : D(x) = 0\}$ forms a subring of R.

A valued D-field is a valued field (K, v) given with a D-ring structure for which $v(e) \geq 0$ and $v(Dx) \geq v(x)$ holds universally.

If (R, D, e) is a D-ring, the ring of D-polynomials over R, is $R\langle X\rangle_D$. As a ring, $R\langle X\rangle_D$ is the polynomial ring over R in the countably many indeterminates $\{D^j X\}_{j\in\omega}$. The ring of D-polynomials has a unique D-structure extending that on R with $D(D^j X) = D^{j+1}X$. Given an element $P(X)$ of $R\langle X\rangle_D$ may be written in the form $P(X) = F(X, \ldots, D^d X)$ for some $F(X_0, \ldots, X_d) \in R[X_0, \ldots, X_d]$. We define $\frac{\partial}{\partial X_i} P = (\frac{\partial}{\partial X_i} F)(X, \ldots, D^d X)$. The order of a differential polynomial $P(X)$, $\mathrm{ord}P$, is $-\infty$ if $P \in K$ and is the least d such that $P \in K[X, \ldots, D^d X]$ otherwise. If $P(X) = F(X, \ldots, D^d X)$ with $d = \mathrm{ord}P$ then the degree of P is $\deg_{X_d} F$. The total degee of P is the sequence $(\deg_{X_i} F)_{i\in\omega}$.

3 Leading Terms

If (K, v) is a valued field with ring of integers $\mathcal{O}_K$, then the ideals in $\mathcal{O}_{K,v}$ correspond to cuts in $\Gamma_{K,v,\geq 0} := \{v(x) : x \in \mathcal{O}_K\}$ via $I \Leftrightarrow v(I) := \{v(a) : a \in I\}$. For an element $\gamma \in \Gamma_{K,v,\geq 0}$ write I_γ for $\{x \in \mathcal{O}_K : v(x) > \gamma\}$. Note that $I_0 = \mathfrak{m}_K$, the maximal ideal of $\mathcal{O}_K$.

If $I \subset \mathcal{O}_K$ is a proper ideal, then because $\mathcal{O}_K$ is a local ring, $1 + I$ is a subgroup of $K^\times$ under multiplication. We denote by K_I the factor group $K^\times/(1+$

I). The structure K_I has been defined merely as a group, but it inherits considerable structure from K.

The valuation v on K is defined by the exact sequence

$$1 \longrightarrow O_K^\times \longrightarrow K^\times \xrightarrow{v} vK \longrightarrow 1$$

Since $1 + I \leq \mathcal{O}_K^\times$, the valuation descends to K_I giving the following exact sequence.

$$1 \longrightarrow (\mathcal{O}_K/I)^\times \longrightarrow K_I \xrightarrow{v} vK \longrightarrow 1$$

If $I \subseteq J \subset \mathcal{O}_K$ are proper ideals, then we have a natural projection map $\pi_{I,J} : K_I \to K_J$. As $K_{(0)} \cong K^\times$, we write $\pi_I : K^\times \to K_I$ for the map $\pi_{(0),I}$.

The ultrametric triangle inequality implies that addition on K leaves a strong trace on K_I. For a pair of ideals $I \subseteq J \subset \mathcal{O}_K$ we define a partial binary operation $+_{I,J}$ on K_I with values in K_J by $z = x +_{I,J} y$ if and only if for any $x', y' \in K$ with $\pi_I(x') = x$ and $\pi_I(y') = y$ we have $\pi_J(x' + y') = z$. In the case that $I = \{x \in \mathcal{O}_K : v(x) > \epsilon\}$ and $J = \{x \in \mathcal{O}_K : v(x) > \delta\}$ with $\delta \leq \epsilon$, then $+_{I,J}$ is defined at (x, y) if and only if for any (some) $(x', y') \in K^2$ with $\pi_I(x') = x$ and $\pi_I(y') = y$ we have $v(x' + y') + \delta \leq \min\{v(x') + \epsilon, v(y') + \epsilon\}$. In particular, if $I = J = \mathfrak{m}_K$, then $\pi_I(x) +_{I,I} \pi_I(y)$ is defined if and only if $v(x + y) = \min\{v(x), v(y)\}$.

Other operations on K induce structure on K_I. For instance, if $\sigma : K \to K$ is a field endomorphism which maps I back to itself, then σ induces an endomorphism of K. In particular, if σ is an automorphism preserving the valuation ($v(x) = v(\sigma(x))$ for $x \in K^\times$), then σ induces a valuation preserving automorphism of K_I.

Recall that a D-ring is a commutative ring R given together with a fixed element $e \in R$ and an additive map $D : R \to R$ satisfying $D(1) = 0$ and the twisted Leibniz rule $D(ab) = aD(b) + bD(a) + eD(a)D(b)$. A valued D-field is a valued field (K, v) given with the structure of a D-ring so that $v(e) \geq 0$ and $v(Dx) \geq v(x)$ for all nonzero $x \in K$.

If $D : K \to K$ gives K the structure of a valued D-field, then D induces a partial function on K_I. As with the partial addition, take $I \subseteq J \subset \mathcal{O}_K$. We define the partial function $D_{I,J} : K_I \to K_J$ by $D_{I,J}\pi_I(x) := \pi_J(D(x))$ if this assignment is well defined. If $I = \{x \in \mathcal{O}_K : v(x) > \delta\}$ and $J = \{x \in \mathcal{O}_K : v(x) > \epsilon\}$, then $D_{I,J}(\pi_I(x))$ is defined if and only if $v(Dx) \leq v(x) + \delta - \epsilon$.

Of course, other structure from K also descends to the leading terms, but we shall need nothing beyond what we have already lain out.

Let $\mathcal{I}$ be a set of ideals in $\mathcal{O}_K$. A system of angular component functions for the pure valued field K relative to $\mathcal{I}$ is a family of group homomorphisms $\text{æ}_I : K_I \to (\mathcal{O}_K/I)^\times$ which are sections of the inclusions defining the valuation on K_I and which are compatible in the sense that if $I \subseteq J$ are two elements of $\mathcal{I}$, then $\pi_{I,J} \circ \text{æ}_I = \text{æ}_J \circ \pi_{I,J}$. The function æ_I is called an angular component function of level I. We abuse notation and write æ_I for $\text{æ}_I \circ \pi_I$ as well. Angular component functions respect addition weakly. More precisely, we have the following lemma.

Lemma 3.1 *If (K, v) is a valued field, $I \subset \mathcal{O}_K$ is a proper ideal, $\text{æ}_I : K^\times \to (\mathcal{O}_K/I)^\times$ is an angular component function of level I, and $x, y \in K^\times$ with $v(x+y) = v(x) = v(y)$, then $\text{æ}_I(x + y) = \text{æ}_I(x) + \text{æ}_I(y)$.*

Proof Since æ_I is a section, if $x \in K^\times$ and $\pi_I(x) \in (\mathcal{O}_K/I)^\times$, then we have $\text{æ}_I(x) = \pi_I(x)$. From this observation we compute:

$\text{ac}_I(x+y) = \text{ac}_I(x)\text{ac}_I(1+\frac{y}{x}) = \text{ac}_I(x)\pi_I(1+\frac{y}{x}) = \text{ac}_I(x) + \text{ac}_I(x)\pi_I(\frac{y}{x}) = \text{ac}_I(x) + \text{ac}_I(x)\text{ac}_I(\frac{y}{x}) = \text{ac}_I(x) + \text{ac}_I(y)$ □

However, if $v(x+y) \neq v(x)$ and $v(x+y) \neq v(y)$, then we can say nothing about $\text{ac}_I(x+y)$ in terms of $\text{ac}_I(x)$ and $\text{ac}_I(y)$.

On occasion, we will need to use the fact that angular component functions preserve sums of arbitrarily many terms.

Lemma 3.2 *Let (K, v) be a valued field of residue characteristic zero. Let n be a positive integer. If $x_1, \ldots, x_{n+1} \in K^\times$ all have the same valuation γ, then $\gamma = v(\sum_{j\neq i} x_j)$ for some i.*

Proof Dividing by x_1 we may assume that $v(x_i) = 0$ for each i. Thus, it suffices to show that if k is a field of characteristic zero and $S \subseteq k^\times$ is a finite set of non-zero elements of size $n+1$, then for some $s \in S$ the sum $\sum_{t\neq s} t \neq 0$.

We prove this by induction on n.

The case of $n = 1$ is trivial.

In the case of $n = 2$, observe that $x_1 + x_2 = 0 = x_1 + x_3$ implies that $x_2 = x_3$ which combined with $x_2 + x_3 = 0$ gives $2x_2 = 0$. As the characteristic is not two, this gives $x_2 = 0$ contrary to our hypotheses.

Now for the inductive case of $n = m+1$. By induction, there is some subset of size m of the first $m+1$ elements with non-zero sum. Without loss of generality, we may assume that $\sum_{i=1}^m x_i \neq 0$. If $\sum_{i=1}^{m+1} x_i = 0$ and $(\sum_{i=1}^m x_i) + x_{n+1} = 0$, then we have $x_n = x_{n+1} = -\sum_{i=1}^m x_i$. We may assume that $x_i \neq x_j$ for some i and j, as otherwise $\sum_{i=1}^{m+1} x_i = nx_1 \neq 0$. Thus, without loss of generality $x_1 \neq x_n = x_{n+1}$. Applying the above reasoning to $\{x_j : j > 1\}$ we find that either $\sum_{j=2}^{n+1} x_j \neq 0$ or $x_1 = x_i$ for some i (say, $i = 2$) and $\sum_{j=3}^{n+1} x_j = -x_1$. Write $y := \sum_{i=3}^m$. Then, we have $2x_1 + y + x_n = x_1 + y + 2x_n$. This yields $x_1 = x_n$ contrary to our choice of x_1. □

Using Lemma 3.2 we derive a sum formula for angular component functions.

Lemma 3.3 *Let K be a valued field of equicharacteristic zero with an angular component funciton ac_I with respect to some ideal $I \subset \mathcal{O}_K$. If $x_1, \ldots, x_n \in K^\times$ and $v(\sum_{j=1}^n x_j) = v(x_1) = \cdots = v(x_n)$, then for some permutation $\pi \in S_n$ we have $\text{ac}_I(\sum_{j=1}^n x_j) = \sum_{i=1}^n x_{\pi(i)}$.*

Proof We work by induction on n. The case of $n = 1$ is trivial.

Take the case of $n+1$. Using Lemma 3.2 permute $x_1, \ldots, x_{n+1}$ so that $v(\sum_{i=1}^j x_i) = v(x_1)$ for each $j \leq n$.

By induction and Lemma 3.1 we have $\text{ac}_I(\sum_{i=1}^{n+1} x_i) = \text{ac}_I(\sum_{i=1}^n x_i + x_{n+1}) = \text{ac}_I(\sum_{i=1}^n x_i) + \text{ac}_I(x_{n+1}) = \sum_{i=1}^n \text{ac}_I(x_i) + \text{ac}_I(x_{n+1}) = \sum_{i=1}^{n+1} \text{ac}_I(x_i)$. □

Remark 3.4 If we adopt the convention that $\text{ac}_I(x) + \text{ac}_I(y) = 0$ if $v(x+y) > v(x) = v(y)$, then the choice of π in Lemma 3.3 is unnecessary. Moreover, with this convention in place the result is valid without regard to the residue characteristic.

If (K, v, D, e) is a valued D-field, then recall that $\sigma : K \to K$ defined by $\sigma(x) := eD(x) + x$ is an endomorphism of K. If $v(e) > 0$, then σ is necessarily an automorphism and $v(x) = v(\sigma(x))$ holds for all $x \in K^\times$. However, if $v(e) = 0$, the endomorphism σ need be neither surjective nor valuation preserving.

Convention 3.5 *From now on we include as part of the definition of a valued D-field that σ is a valuation preserving automorphism.*

If K is a valued D-field and $\mathcal{I}$ is a family of ideals in $\mathcal{O}_K$, then a system of D-field angular component functions for $\mathcal{I}$ is defined to be a system of angular component functions for $\mathcal{I}$ in the sense of pure valued fields which respect the D-structure. That is, $\text{ac}_J(D_{I,J}(x)) = \pi_{I,J}(D\text{ac}_I(x))$ for x with $v(Dx) = v(x)$ and $\text{ac}_I(\sigma(x)) = \sigma(\text{ac}_I(x))$ for all x.

From an angular component function $\text{ac}_I : K_I \to (\mathcal{O}_K/I)^\times$ we obtain a section of the valuation $\chi_I : \Gamma_K \to K_I$ by the formula $\chi_I(\gamma) := \frac{x}{\text{ac}_I(x)}$ for any choice of $x \in K_I$ with $v(x) = \gamma$.

Lemma 3.6 *Let $(K, 0, 1, e, +, \cdot, D, v)$ be a valued D-field, $I \subset \mathcal{O}_K$ a proper ideal, $\text{ac}_I : K_I \to (\mathcal{O}_K/I)^\times$ an angular component function of level I, and $\chi_I : \Gamma_K \to K_I$ the corresponding section on the value group. Then the angular component function ac_I respects the D structure if the range of χ_I is contained in the set of D-constants, the set $\{x \in K_I : (\forall x' \in K^\times)\pi(x') = x \to \frac{Dx'}{x'} \in I\}$. The converse is true as long as D is non-trivial on the residue field.*

Proof We show first that ac_I respect the D-structure if χ_I takes values in the D-constants. Let $x \in K_I$ and suppose that $D_{I,I}(x)$ is defined. Write $x = \text{ac}_I(x)\chi_I(v(x))$. Let $a, b \in K^\times$ with $\pi_I(a) = \text{ac}_I(x)$ and $\pi_I(b) = \chi_I(v(x))$. Then by the twisted Leibniz rule we have $D(ab) = aD(b) + \sigma(b)D(a)$. As χ_I takes values in the D-constants (modulo $1 + I$), we have $D(b) \in bI$ and $\sigma(b) \in b(1 + I)$. So we have $D(ab) = bD(a) + bD(a)i + abi'$ for some $i, i' \in I$. As K is a valued D-field, we have $v(bD(a)) \geq v(ab)$. Thus, the hypothesis that $v(x) = v(Dx)$ implies that $v(Da) = v(a)$. Thus, we have $D(ab) \equiv bD(a)(1+I)$. Applying π_I, we have $D(x) = \chi_I(v(x))D(\text{ac}_I(x))$. As $v(x) = v(D(x))$, we have $\chi_I(D(x)) = \chi_I(v(D(x))D(\text{ac}_I(x))$. As generally we have $\text{ac}_I(y) = \frac{y}{\chi_I(v(y))}$, we conclude that $D(\text{ac}_I(x)) = \text{ac}_I(Dx)$.

For the other implication, let $\gamma \in \Gamma_K$.

The hypothesis that D is non-trivial on the residue field implies that we can find some $x \in K^\times$ with $v(x) = \gamma = v(Dx)$. That is, choose any y with $v(y) = \gamma$. If y does not already work, then look for some α with $v(\alpha) = 0$ and $\gamma = v(D(\alpha y)) = v(D(\alpha)y+\sigma(\alpha)D(y))$. As $v(D(y)) > \gamma$ and $v(\alpha) = 0$, we achieve this is $v(D(\alpha)) = 0$.

For this choice of x, write $x = \text{ac}_I(x)\chi_I(\gamma)$. As $v(x) = v(Dx)$ we have $D(x) = \text{ac}_I(Dx)\chi_I(\gamma)$ and $D(x) = D(\text{ac}_I(x))\chi_I(\gamma) + \sigma(\text{ac}_I(x))D(\chi_I(\gamma)) = \text{ac}_I(Dx)\chi_I(\gamma) + \sigma(\text{ac}_I(x))D(\chi_I(\gamma))$. As $v(\sigma(\text{ac}_I(x))) = v(\text{ac}_I(x)) = 0$, we conclude that $\frac{D(\chi_I(\gamma))}{\chi_I(\gamma)} \in I$ as claimed. □

4 Languages

We treat valued D-fields as many-sorted structures having a basic sort K for the valued field itself and other sorts for the residue field, k; the value group, Γ; and, perhaps, also some other sorts for leading terms and residue rings. All these sorts are interpretable in the basic sort; so nothing is lost by working in a one-sorted structure and treating the other sorts as merely interpreted. However, doing this requires one to relatize the quantifier elimination statements of our main theorem.

The language of D-rings, $\mathcal{L}_{\text{D-ring}} := \mathcal{L}(+, \cdot, -, D, 0, 1, e)$, is the expansion of the language of rings by a new unary function symbol D and a new constant symbol e. As mentioned in the previous section, if (R, D, e) is a D-ring, then the function $\sigma : R \to R$ defined by $\sigma(x) := eD(x)+x$ is a ring endomorphism. We insist that σ is actually an automorphism so that the $\mathcal{L}_{\mathrm{D-ring}}$-structures we consider have natural expansions to $\mathcal{L}_{\mathrm{D-ring}}(\sigma, \sigma^{-1})$.

We regard K and k as $\mathcal{L}_{\mathrm{D-ring}}(\sigma, \sigma^{-1})$-structures and Γ as an $\mathcal{L}(+, -, \leq, 0)$-structure where the non-logical symbols have their usual meaning. These sorts are connected by the valuation $v : K \to \Gamma \cup \{\infty\}$ and residue map $\pi : K \to k \cup \{\infty\}$, where we have adjoined a new symbol "∞" not belonging to any particular sort to ensure that π and v are total. We call the language described above the language of valued D-fields, $\mathcal{L}_{\mathrm{vdf}}$.

One could consider the language of valued D-fields augmented by arbitrarily complicated systems of angular component functions or leading term sorts. However, we need only multiplicative congruences modulo the 1-units in pure characteristic zero and with respect to the family of ideals $\{I_n := \{x \in \mathcal{O}_K : v(x) > nv(p)\}\}_{n \in \omega}$ in mixed characteristic.

We first define the languages appropriate to leading term structures. The language involving just leading terms of level $\mathfrak{m}_K$ is obtained from $\mathcal{L}_{\mathrm{vdf}}$ by adjoining a new sort K_0 to be treated as an $\mathcal{L}_{\mathrm{D-ring}}(\sigma, \sigma^{-1})$-structure and a new function symbol $\pi_0 : K \to K_0$. Moreover, we extend the function v to be defined on K_0 as well and treat k as a substructure of K_0. We will interpret K_0 as $K_{\mathfrak{m}} \cup \{0\}$. The inclusion $k \hookrightarrow K_0$ comes from the valuation exact sequence on $K_{\mathfrak{m}}$ together with $0 \mapsto 0$. We call this language $\mathcal{L}_{\mathrm{vdf}}^{\mathrm{lt},0}$.

The mixed characteristic leading term language is more complicated than the pure characteristic zero leading term language. We add new sorts K_n to be treated as $\mathcal{L}_{\mathrm{D-ring}}(\sigma, \sigma^{-1})$-structures and function symbols $\pi_n : K \to K_n$, $\pi_{m,n} : K_m \to K_n$, $D_{m,n} : K_m \to K_n$, and $+_{m,n} : K_m^2 \to K_n$. We identify k with K_0 as in $\mathcal{L}_{\mathrm{vdf}}^{\mathrm{lt},0}$ and we extend v to each K_m. Since we will interpret K_m as K_{I_m}, we no longer insist that $+$ and D be total functions. Perhaps, one might like to think of the new sorts as simply $\mathcal{L}(\cdot, \sigma, \sigma^{-1}, 1, e)$-structures and the partial functions $+_{n,m}$ and $D_{n,m}$ as merely relations. Instead, we regard the value of any of our partial functions applied to a point where it is not defined to be 0. This language is denoted by $\mathcal{L}_{\mathrm{vdf}}^{\mathrm{lt},\omega}$.

The language appropriate for considering a pure characteristic zero valued D-field with angular components is $\mathcal{L}_{\mathrm{vdf}}^{\mathrm{ac},0} := \mathcal{L}_{\mathrm{vdf}}^{\mathrm{lt},0}(\mathrm{ac}_0)$ where $\mathrm{ac}_0 : K_0 \to k$ is a unary function symbol which will be interpreted as an angular component function of level $\mathfrak{m}_K$ (and $\mathrm{ac}_0(0) := 0$).

The language appropriate for mixed characteristic valued D-fields with angular components is $\mathcal{L}_{\mathrm{vdf}}^{\mathrm{ac},\omega}$, a definitional expansion of $\mathcal{L}_{\mathrm{vdf}}^{\mathrm{lt},\omega}(\{\mathrm{ac}_n\}_{n \in \omega})$ where ac_n is a unary function symbol on K_n with values in K_n. We will interpret $\{\mathrm{ac}_n\}_{n \in \omega}$ as a system of angular component functions for the ideals $\{I_n : n \in \omega\}$. We add new sorts R_n, one for each $n \in \omega$. We interpret $R_n(K)$ as $\mathcal{O}_K / I_n$ embedded in $K_n \cup \{0\}$. The partial addition functions on K_n induce the total functions on $R_n(K)$.

5 Relative completeness and quantifier elimination

In the original version of this paper [7], the angular component and leading term languages were considered in parallel. As noted in the introduction, while the angular component language is more transparent, the leading term structure is already interpretable in the language of pure valued fields and is therefore closer to the basic language. In this section we observe that completeness and relative quantifier elimination for the leading term languages follow from the corresponding theorems for the angular component language.

Definition 5.1 Let $\mathcal{L}$ be a many-sorted first order language. Let Σ be a set of $\mathcal{L}$-sorts. We say that Σ is closed if for any $S_1, \ldots, S_n \in \Sigma$, sort S, and non-constant $\mathcal{L}$-term $t(x_1, \ldots, x_n)$ with domain $S_1 \times \cdots \times S_n$ and range S, we have $S \in \Sigma$.

Remark 5.2 Note that a closed set of sorts is closed under products. So in the definition of closed we could say instead that Σ is closed under products and consider only t unary terms.

Lemma 5.3 *Suppose that $\mathcal{L} \subseteq \mathcal{L}'$ are many-sorted languages having the same sorts. Let Σ be a closed set of sorts. Let T' be a complete $\mathcal{L}'$-theory with $T := T' \restriction \mathcal{L}'$ its restriction to $\mathcal{L}'$. Let $\mathcal{L}''$ be the definitional expansion of $\mathcal{L}$ by basic predicates for each $\mathcal{L}$-definable relation on any sort in Σ and let $T'' \supseteq T$ be the theory of this definitional expansion. Suppose that for each $\mathcal{L}$ formula $\varphi(x)$ there is a quantifier-free $\mathcal{L}'$ formula $\theta(x)$ such that $T' \vdash \varphi(x) \leftrightarrow \theta(x)$. Suppose also that every quantifier-free $\mathcal{L}'$-formula is provably equivalent to a finite boolean combination of quantifier-free $\mathcal{L}$-formulas and formulas of the form $\psi(t(x))$ where $\psi(y)$ is a quantifier-free $\mathcal{L}'$ formula with y ranging over a sort in Σ and t is an $\mathcal{L}$-term. Then T'' eliminates quantifiers in $\mathcal{L}''$.*

Proof Let $\varphi(x) \in \mathcal{L}(x)$ be an $\mathcal{L}$-formula. By our first hypothesis there is a quantifier-free $\mathcal{L}'$ formula $\theta(x)$ such that $T' \vdash \varphi(x) \leftrightarrow \theta(x)$. By our second hypothesis, we may assume that θ is of the form $\vartheta(t(x))$ where $\vartheta(y)$ is a quantifier-free $\mathcal{L}'$ formula with y ranging over a sort in S and t is an $\mathcal{L}'$-term. Let $\zeta(z) := (\exists x)(\varphi(x) \wedge z = t(x))$ considered as a quantifier free formula in $\mathcal{L}''$. So we have $T' \cup T'' \vdash \zeta(t(x)) \leftrightarrow \vartheta(t(x))$. This yields, $T'' \vdash \varphi(x) \leftrightarrow \zeta(t(x))$ which shows that $\varphi(x)$ is equivalent to a quantifier-free $\mathcal{L}''$ formula. $\square$

The hypotheses of the previous lemma were quite strong but the next lemma shows that they hold in many cases.

Lemma 5.4 *Let $\mathcal{L} \subseteq \mathcal{L}'$ be two many-sorted languages having the same sorts. We suppose that $\mathcal{L}' \setminus \mathcal{L}$ consists of new function symbols. We assume also that Σ is a $\mathcal{L}'$-closed set of sorts and that every new function symbol f in $\mathcal{L}' \setminus \mathcal{L}$ has domain $D_f \in \Sigma$ and range $R_f \in \Sigma$. Then every quantifier-free $\mathcal{L}'$-formula is equivalent to a finite boolean combination of quantifier-free $\mathcal{L}$ formulas and $\mathcal{L}'$ formulas of the form $\theta(t(x))$ where $\theta(y)$ is a quantifier-free $\mathcal{L}'$-formula with y ranging over a sort in S and t is an $\mathcal{L}$-term.*

Proof Working by induction on the complexity of $\varphi(x)$ it suffices to consider the case of a basic relation $R(t(x))$ where R is a $\mathcal{L}$-relation and t is an $\mathcal{L}'$ term.

Claim 5.5 *If $s(x)$ is a $\mathcal{L}'$ term with range the sort $S \notin \Sigma$, then there is a $\mathcal{L}$ term $r(x)$ such that $\vdash (\forall x) r(x) = s(x)$.*

Proof of Claim: We work by induction on the complexity of s. Write $s = f \circ t$ where t is term of lower complexity (possibly empty) and f is either a basic $\mathcal{L}'$-symbol or a constant symbol. If f is a constant symbol, then by hypothesis, f is already an $\mathcal{L}$ term and $\vdash s = f$. If f is a not a constant symbol and the sort of the domain of f belongs to Σ, then as Σ is closed, necessarily the range of f is also in Σ. If f is not a constant symbol and its domain sort does not belong to Σ, then f

is an $\mathcal{L}$-term and by induction there is some $\mathcal{L}$-term t' for which $\vdash (\forall x)t(x) = t'(x)$. Thus, we may take $r = f \circ t'$. $\dashv$

Thus, if the range of t is not a sort in Σ, then $R(t(x))$ is provably equivalent to a quantifier-free $\mathcal{L}$ formula.

Claim 5.6 *If $s(x)$ is a $\mathcal{L}'$ term with range the sort $S \in \Sigma$, then there are a $\mathcal{L}$ term $t(x)$ and a $\mathcal{L}'$ term $r(y)$ with y ranging over some sort in Σ and $\vdash (\forall x)s(x) = r(t(x))$.*

Proof of Claim: Again we work by induction on the complexity of s, but this time from the right. Write $s = f \circ s'$ with f a basic function or constant symbol. We allow the case that s' is the empty term, interpreted as an identity function. If the range of s' is a sort in Σ, then by induction we may write $s' = r' \circ t'$ with t' an $\mathcal{L}$-term and r' an $\mathcal{L}'$-term with domain a sort in Σ. We can then take $r := f \circ r'$ and $t := t'$. If the range of s' is not a sort in Σ, then by Claim 5.5 s' is provably equal to an $\mathcal{L}$-term s''. As every new symbol in $\mathcal{L}' \setminus \mathcal{L}$ is a function symbol with domain a sort in Σ, necessarily f is an $\mathcal{L}$-term. Thus, we may take $t := f \circ s''$ and r the identity function on S in this case. $\dashv$

Thus, in the case that the range of t is a sort in Σ we may write $t = r \circ s$ with s an $\mathcal{L}$-term with range a sort in Σ and r an $\mathcal{L}'$-term. We then set $\vartheta(y) := R(r(y))$ and s plays the role of t in the statement of the lemma. $\square$

The content of Lemma 5.3 is best expressed in terms of relative quantifier elimination.

Definition 5.7 Let $\mathcal{L}$ be a many sorted language and Σ a closed set of $\mathcal{L}$-sorts. Let T be a $\mathcal{L}$-theory. Let $\mathcal{L}'$ be the definitional expansion of $\mathcal{L}$ (and T' the definitional expansion of T to $\mathcal{L}'$) by predicates for all the definable relations on sorts in Σ. If T' eliminates quantifiers in $\mathcal{L}'$, then we say that T eliminates quantifiers relative to Σ.

Corollary 5.8 *If $\mathcal{L}$ is an expansion of $\mathcal{L}_{\mathrm{vdf}}$ by predicates on the leading terms and the theory of some valued D-field admits quantifier elimination in the expansion of $\mathcal{L}$ be angular component functions, then it also eliminates quantifiers in the expansion of $\mathcal{L}$ relative to the leading terms.*

Proof The new symbols in the angular component language are all function symbols on leading terms, so we may apply Lemma 5.3. $\square$

Besides the notion of relative quantifier elimination, we also have the notion of relative completeness.

Definition 5.9 Let $\mathcal{L}$ be a many sorted language and Σ a closed set of $\mathcal{L}$ sorts. Let T be a $\mathcal{L}$-theory. We say that T is complete relative to Σ if for any $\mathcal{M} \models T$, the theory $T \cup \mathrm{Th}(\bigcup_{S \in \Sigma} S(\mathcal{M}))$ is complete.

In Corollary 5.8 we allow arbitrary expansions of $\mathcal{L}_{\mathrm{vdf}}$ by predicates on the leading terms. We express this persistence of relative quantifier elimination as resplendent relative quantifier elimination relative to Σ. Likewise, we have the notion of resplendent relative completeness relative to Σ.

6 Axioms

The class of valued D-fields considered in this paper is larger than that of [6] due to the relaxation of three conditions on the residue field. First, we no longer insist that the residue characteristic is zero. Secondly, we now allow for the possibility of $v(e) = 0$. Finally, we do not demand that the residue field be closed under roots. We require only that the residue field be linearly D-closed. As shown in [6], this is an intrinsic property of D-henselian fields. We also insist on generic characteristic zero. It ought to be possible to relax this last condition as well, but given the current state of knowledge about the model theory of pure valued fields of positive characteristic, I expect this would take significantly new methods.

The axioms for valued D-fields given in [6] do not suffice for the more general valued D-fields we consider here. Since the language has been expanded and the conditions on the residue field relaxed, we need to modify the axioms for valued D-fields. In stating the axioms, we give the axioms for valued D-fields in the basic language of valued D-fields, $\mathcal{L}_{\mathrm{vdf}}$.

The first three axioms describe valued D-fields.

Axiom 1 *K is D-field, k is a D-field and Γ is an ordered abelian group. [NB: we require a D-field to be an $\mathcal{L}_{\mathrm{D-ring}}(\sigma, \sigma^{-1})$-structure in which σ is an automorphism with inverse σ^{-1} satisfying the equation $\sigma = eD + \mathrm{id}$.]*

Axiom 2 *The inequality $v(Dx) \geq v(x)$ holds universally as does the equality $v(x) = v(\sigma(x))$.*

Axiom 3 *K is a valued field with value group a subgroup of Γ via v and residue field a subfield of k via π. The map π restricted to $\mathcal{O}_K$ is a map of $\mathcal{L}_{\mathrm{D-ring}}(\sigma, \sigma^{-1})$-structures.*

The next there axioms finish the description of D-henselian fields.

Axiom 4 *The maps π and v are surjective.*

Axiom 5 *K has enough constants: $v((K^D)^\times) = \Gamma$. That is, for any $\gamma \in \Gamma$ there is some $x \in K$ with $Dx = 0$ and $v(x) = \gamma$.*

Axiom 6 *D-Hensel's Lemma: If $P(X) \in \mathcal{O}_K\langle X \rangle_D$ and $a \in \mathcal{O}_K$ with $v(P(a)) > 0 = v(\frac{\partial}{\partial X_i} P(a))$ for some non-negative integer i, then there is some $b \in \mathcal{O}_K$ with $P(b) = 0$ and $v(a - b) > 0$.*

While the axioms are given in $\mathcal{L}_{\mathrm{vdf}}$, we have natural expansions to the languages $\mathcal{L}_{\mathrm{vdf}}^{\mathrm{lt},0}$ and $\mathcal{L}_{\mathrm{vdf}}^{\mathrm{lt},\omega}$.

A valued D-field with an angular component function (of level zero) is a valued D-field given together with an angular component function $\text{æ}_0 : K_0 \to k$ of level $\mathfrak{m}$. Likewise, a valued D-field with a system of angular component functions is an expansion of a valued D-field to the language $\mathcal{L}_{\mathrm{vdf}}^{\mathrm{ac},\omega}$ in which the angular component function symbols are interpreted as a system of angular components for $\{I_n := \{x \in \mathcal{O}_K : v(x) > nv(p)\}\}_{n \in \omega}$.

Remark 6.1 In [6] the axioms for a valued D-field are given relative to a fixed theory of the residue field and value group. We present a semantic version of that completeness result below. The syntactic form is somewhat more complicated for mixed characteristic valued D- fields. In particular, it does not suffice to specify the theory of the residue field and of the value group in order to give a completion of the theory of D-henselian fields.

Remark 6.2 Proposition 5.3 of [6] applies to the modified axioms for D-henselian fields also. Thus, if K is a D-henselian field, its residue field, k, is linearly D-closed.

We can now state our main theorem.

Theorem 6.3 *The theory of D-henselian fields of characteristic zero in the language $\mathcal{L}_{\mathrm{vdf}}^{\mathrm{ac},\omega}$ together with a consistent atomic diagram is resplendently complete and resplendently eliminates quantifiers relative to the value group and the residue rings R_n ($n \in \omega$).*

If we restrict to pure characteristic zero D-henselian fields, then Theorem 6.3 may be stated with $\mathcal{L}_{\mathrm{vdf}}^{\mathrm{ac},0}$ in place of $\mathcal{L}_{\mathrm{vdf}}^{\mathrm{ac},\omega}$.

Theorem 6.4 *The theory of D-henselian fields of pure characteristic zero in the language of $\mathcal{L}_{\mathrm{vdf}}^{\mathrm{ac},0}$ together with a consistent atomic diagram is respelendently complete and resplendently eliminates quantifiers relative to the value group and the residue field.*

Bearing in mind the results of Section 5 we see that $\mathcal{L}_{\mathrm{vdf}}^{\mathrm{ac},\omega}$ may be replaced by $\mathcal{L}_{\mathrm{vdf}}^{\mathrm{lt},\omega}$ relative to the leading terms. In Section 10 we discuss some other cases of interest in which $\mathcal{L}_{\mathrm{vdf}}^{\mathrm{lt},\omega}$ may be replaced by simpler languages.

7 Tests for completeness and quantifier elimination

Our proof of Theorem 6.3 passes through routine variants of standard model-theoretic tests for completeness and quantifier elimination. In [6] an extension of partial isomorphisms test was used to demonstrate completeness and quantifier elimination. We use the same test here, but we also need another version which requires some set theoretic hypotheses.

Recall that if T is a complete theory in a first-order language $\mathcal{L}$ and $\kappa \geq |\mathcal{L}|$ is a cardinal, then T has at most one saturated model of cardinality κ. Of course, if $\mathcal{M}$ and $\mathcal{N}$ are two isomorphic $\mathcal{L}$-structures, then they have the same theory. These observations give a test for completeness of a theory.

Test 7.1 *Let T be a theory in a first-order language $\mathcal{L}$. Suppose that T has no finite models and that each consistent completion of T has saturated models in each cardinality $\kappa > |\mathcal{L}|$. Then T is complete if and only for any two saturated models of T of the same cardinality are isomorphic.*

Test 7.1 includes the extraneous hypothesis that the completions of T have saturated models. In general, one needs to know T very well or make some set theoretic hypotheses beyond ZFC (GCH, for example) in order to verify this hypothesis. However, it is possible to make these hypotheses simply for the purpose of the test and then conclude unconditionally that T is complete. That is, the assertion that a given theory in a countable language is complete is absolute (does not depend on the model of set theory) and the completeness of an arbitrary theory is equivalent to the completeness of its (or really, the set of its consequences) restrictions to countable sublanguages.

Test 7.2 *Let T be a theory in a first-order language $\mathcal{L}$. Suppose that T has no finite models. Then the following are equivalent.*

- *T is complete.*

- *In any model of set theory in which GCH holds, if $\mathcal{M} \models T$ and $\mathcal{N} \models T$ and $|\mathcal{M}| = |\mathcal{N}| > |\mathcal{L}|$, then $\mathcal{M} \cong \mathcal{N}$.*
- *In some model of set theory in which GCH holds and there is some cardinal $\kappa > |\mathcal{L}|$ such that any two saturated models of T of cardinality κ are isomorphic.*

We have similar tests for quantifier elimination. As with completeness, the property of a theory admitting quantifier elimination is absolute. Thus, we can work in model of set theory in which saturated models exist abundantly. In such a universe, quantifier elimination for a complete theory is equivalent to condition that every partial automorphism (with a small domain) of a saturated model extends to an automorphism. Using a back-and-forth, one finds a more constructive version of this latter condition in terms of extending a partial automorphism to one new element.

Test 7.3 *Let T be a complete theory in a first-order language $\mathcal{L}$. The following are equivalent.*

- *T eliminates quantifiers: For any formula $\varphi(x_1, \ldots, x_n) \in \mathcal{L}(x_1, \ldots, x_n)$ there is a quantifier-free formula $\vartheta(x_1, \ldots, x_n) \in \mathcal{L}(x_1, \ldots, x_n)$ such that $T \vdash (\forall x_1, \ldots, x_n)\varphi(x_1, \ldots, x_n) \leftrightarrow \vartheta(x_1, \ldots, x_n)$.*
- *In any model of set theory in which GCH holds, if $\mathcal{M} \models T$ is a saturated model and $f : A \to B$ is an $\mathcal{L}$-isomorphism where $A, B \subseteq \mathcal{M}$ are substructures with $|A| = |B| < |\mathcal{M}|$, then there is an automorphism $\sigma : \mathcal{M} \to \mathcal{M}$ such that $\sigma|_A = f$.*
- *If $\mathcal{M} \models T$ is a $|\mathcal{L}|^+$-saturated model, $A, B \subseteq \mathcal{M}$ are substructures, $f : A \to B$ is an $\mathcal{L}$-isomorphism, $|A| \leq |T|$, and $a \in \mathcal{M}$, then there is an extension of f to an $\mathcal{L}$-embedding of the structure generated by a over A into $\mathcal{M}$.*

Since the tests for quantifier elimination and for completeness are so similar, we can combine them into a single extension of partial isomorphism test.

Test 7.4 *Let T be a theory in a first-order language $\mathcal{L}$. Suppose that T has no finite models, that T is complete with respect to the atomic theory (that is, for each atomic sentence ψ of $\mathcal{L}$ either $T \vdash \psi$ or $T \vdash \neg\psi$), and that $\mathcal{L}$ has at least one constant symbol. Then the following are equivalent.*

- *T is complete and eliminates quantifiers.*
- *In any model of set theory in which GCH holds, if $\mathcal{M}, \mathcal{N} \models T$ are saturated models of T of the same cardinality, $A \subseteq \mathcal{M}$ and $B \subseteq \mathcal{N}$ are substructures of cardinality strictly less than that of $\mathcal{M}$, and $f : A \to B$ is an $\mathcal{L}$-isomorphism, then there is an $\mathcal{L}$-isomorphism $g : \mathcal{M} \to \mathcal{N}$ such that $g|_A = f$.*
- *If $\mathcal{M}, \mathcal{N} \models T$ are $|T|^+$-saturated, $A \subseteq \mathcal{M}$ is a substructure of cardinality at most $|T|^+$, $f : A \hookrightarrow \mathcal{N}$ is an $\mathcal{L}$-embedding, and $a \in \mathcal{M}$, then there is a substructure A' of $\mathcal{M}$ containing A and a and extension of f to an $\mathcal{L}$-embedding $g : A' \hookrightarrow \mathcal{L}$.*
- *If $\mathcal{L}' \subseteq \mathcal{L}$, $\mathcal{M}, \mathcal{N} \models T|_{\mathcal{L}'}$ are models of the restriction of (the set of consequences of T) to $\mathcal{L}'$, $A \subseteq \mathcal{M}$ is a countable substructure, $f : A \hookrightarrow \mathcal{N}$ is an $\mathcal{L}'$-embedding, and $a \in \mathcal{M}$, then there is an elementary extension $\mathcal{N}' \succeq \mathcal{N}$ of $\mathcal{N}$, a substructure $A' \subseteq \mathcal{M}$ with $A \cup \{a\} \subseteq A'$, and extension of f to an $\mathcal{L}'$-embedding $f' : A' \to \mathcal{N}'$.*

It is this test that we use to prove Theorem 6.3. We take for $\mathcal{L}$ some expansion of $\mathcal{L}_{\mathrm{vdf}}^{\mathrm{lt},\omega}$ or $\mathcal{L}_{\mathrm{vdf}}^{\mathrm{ac},\omega}$ by predicates on the leading terms. We take for T the theory of D-henselian fields of characteristic zero together with a consistent atomic diagram and a complete theory in the restriction of $\mathcal{L}$ to the leading terms. It is worth noting that while in the definition of resplendent relative completeness and resplendent relative quantifier elimination arbitrary expansions of the base language are permitted, it suffices to consider only countable languages.

8 Reduction to Pure Characteristic Zero

We have stated on many occasions that the theory of the relative Frobenius motivates our study of more general D-henselian fields. In this section we show that the relative Frobenius fits into the framework of D-henselian fields and that standard coarsening arguments permit us to reduce our general problem to the case of pure characteristic zero D-henselian fields.

We start with some observations about the relation between linear D-closedness, linear difference closedness, and D-henselianness.

Lemma 8.1 *Let (K, σ) be a difference field. Let $e \in K^{\times}$ be a unit. Give K the structure of a D-field by setting $D(x) := \frac{\sigma(x)-x}{e}$. Then, K is linearly D-closed if and only if K is linearly difference closed (ie for any nonzero polynomial $\sum_{j=0}^{m} a_j X^j \in K[X]$ the linear difference operator $\sum_{j=0}^{m} a_j \sigma^j$ is surjective on K).*

Proof:
An easy induction shows for any natural number n that $D^n = (\prod_{j=0}^{n-1} \sigma^j(e))\sigma^n +$ {lower order terms} and $\sigma^n = (\prod_{j=0}^{n-1} \sigma^j(e)^{-1})D^n +$ {lower order terms}. Thus, for any sequence $(a_0, \dots, a_m) \in K^{m+1}$ we have

$$\begin{aligned}\sum_{j=0}^{m} a_j D^j &= a_m(\prod_{j=0}^{m-1} \sigma^j(e))\sigma^m + \text{lower order terms} \\ \sum_{j=0}^{m} a_j \sigma^j &= a_m(\prod_{j=0}^{m-1} \sigma^j(e)^{-1})D^m + \text{lower order terms}\end{aligned}$$

So, if K is linearly difference closed and $a_m \neq 0$, then the operator $\sum_{j=0}^{m} a_j D^j$ is equal to a difference operator having a non-zero leading co-efficient and is therefore surjective. Likewise, linear D-closedness implies linear difference closedness. $\dashv$

Corollary 8.2 *If K is a field of characteristic $p > 0$ with no extensions of degree p, $e \in K^{\times}$, $n \in \mathbb{Z} \setminus \{0\}$ and $D : K \to K$ is defined by $D(x) := \frac{x^{p^n}-x}{e}$, then K is linearly D-closed.*

Lemma 8.3 *If (K, v, D) is D-henselian, then K is linearly D-closed.*

Proof: Let $L = \sum_{j=0}^{n} a_j D^j \in K[D]$ be a non-zero linear D-operator. Let $\alpha \in K$. If $\alpha = 0$, then $L(0) = 0 = \alpha$ so that we may assume that $\alpha \neq 0$. Let $\epsilon \in K$ with $D\epsilon = 0$ and $v(\epsilon) = \max\{v(\alpha) - v(a_j)\}$. Let $Q(Y) := \sum_{j=0}^{n} \frac{a_j \epsilon}{\alpha} D^j(Y) - 1$. By our choice of ϵ, $Q(Y) \in \mathcal{O}_K\langle X\rangle_D$ and the reduction of Q, $\pi(Q)$, is a non-constant affine D- polynomial over the residue field of K. Since the residue field of K is linearly D-closed, we can find some $b \in \mathcal{O}_K$ such that $\pi(Q(b)) = 0$. As $\pi(Q)$ is residually affine, $v(\frac{\partial}{\partial X_i} Q(b)) = 0$ for some i. Thus, DHL applies to Q at b and we can find some $a \in \mathcal{O}_K$ with $v(a-b) > 0$ and $Q(a) = 0$. Let $\tilde{a} := a\epsilon$. Then $L(\tilde{a}) = \alpha(Q(a)+1) = \alpha$. Thus, L is surjective on K. $\dashv$

As with pure fields, D-Hensel's lemma takes many forms. In the next lemma we show the equivalence between the version of D-Hensel's lemma already given and an ostensibly stronger version in which the approximate solution is not assumed to be a simple solution in the residue field.

Lemma 8.4 *Let (K, v, D) be a valued D-field with enough constants and linearly D-closed residue field. The following are equivalent.*

1. *For all D-polynomials $P(X) \in \mathcal{O}_K\langle X\rangle$ and elements $a \in \mathcal{O}_K$ such that $v(P(a)) > 0 = \min\{v(\frac{\partial}{\partial X_i} P(a)) : i \in \omega\}$, there is some $b \in \mathcal{O}_K$ with $P(b) = 0$ and $v(a-b) = v(P(a))$.*
2. *Given $P(X) \in \mathcal{O}_K\langle X\rangle$ and $a \in \mathcal{O}_K$ define $\gamma := \min\{v(\frac{\partial}{\partial X_i} P(a)) : i \in \omega\}$. If $v(P(a)) > 2\gamma$, then there is some $b \in \mathcal{O}_K$ with $P(b) = 0$ and $v(a-b) = v(P(a)) - \gamma$.*

Proof: The implication from (2) to (1) is immediate so we concentrate on proving (1) to (2).

Let $P(X) \in \mathcal{O}_K$ and $a \in \mathcal{O}_K$ be given with $\gamma = \min\{v(\frac{\partial}{\partial X_i} P(a)) : i \in \omega\}$ having $v(P(a)) > 2\gamma$.

If $P(a) = 0$, then there is nothing to prove so we assume now that $P(a) \neq 0$.

Let $\epsilon \in K^D$ with $v(\epsilon) = v(P(a)) - \gamma$. Let $Q(Y) := \frac{1}{P(a)} P(a+\epsilon Y)$. I claim that $Q(Y) \in \mathcal{O}_K\langle Y\rangle_D$. To see this, expand $P(a+\epsilon Y) \equiv P(a) + \sum \frac{\partial}{\partial X_i} P(a)\epsilon D^i Y + (\epsilon^2)$. Of course, $P(a)$ divides $P(a)$, so the constant term of $Q(Y)$ is integral. By our choice of ϵ and the definition of γ, $v(\frac{\partial}{\partial X_i} P(a)\epsilon) = v(\frac{\partial}{\partial X_i} P(a)) + v(\epsilon) \geq v(P(a))$. Thus, the linear term of $Q(Y)$ is integral. Finally, we have $v(\epsilon^2) = 2v(P(a)) - 2\gamma = v(P(a)) + (v(P(a)) - 2\gamma) > v(P(a))$

which implies that the higher terms in $Q(Y)$ have co-efficients in the maximal ideal of $\mathcal{O}_K$.

So, not only is $Q(Y)$ integral, but its reduction is a nonzero affine D-polynomial. As the residue field is linearly D-closed, there is some $b \in K$ such that $v(Q(b)) > 0$ and $v(b) = 0$. By DHL in the original form, there is some $c \in \mathcal{O}_K$ such that $Q(c) = 0$ and $v(c-b) = v(Q(b))$. Set $d := a + \epsilon c$. Then $P(d) = 0$ and $v(a-d) = v(\epsilon) + v(c) = v(\epsilon) = v(P(a)) - \gamma$. $\dashv$

Lemma 8.5 *If (K, v, D) is a D-henselian field and w is a coarsening of the valuation v, then (K, w, D) is also a D-henselian field.*

Proof: We check the axioms.

Let $\tilde{w} : vK \to wK$ be the homomorphism for which $w = \tilde{w} \circ v$. Note that $\tilde{w}$ is order preserving.

Axiom 1 makes no mention of the valuation so it remains true.

For Axiom 2 let $x \in K$, then by hypothesis $v(Dx) \geq v(x)$ and $v(\sigma(x)) = v(x)$. Applying $\tilde{w}$, we obtain $w(Dx) = \tilde{w} \circ v(Dx) \geq \tilde{w} \circ v(x) = w(x)$ and $w(\sigma(x)) = \tilde{w} \circ v(\sigma(x)) = \tilde{w} \circ v(x) = w(x)$.

Axioms 3 and 4 state merely that the extra sorts are interpreted as the value group and residue field. This does not change upon passage from v to w.

For Axiom 5 let $\gamma \in wK$. Let $\tilde{\gamma} \in vK$ with $\tilde{w}(\tilde{\gamma}) = \gamma$. By Axiom 5 for v we can find $\epsilon \in K$ with $v(\epsilon) = \tilde{\gamma}$ and $D\epsilon = 0$. Apply $\tilde{w}$ and we see that $w(\epsilon) = \tilde{w} \circ v(\epsilon) = \tilde{w}(\tilde{\gamma}) = \gamma$.

The only axiom requiring real proof is Axiom 6, D-Hensel's lemma. For this we use the strengthened version of D-Hensel's lemma. If $P(X) \in \mathcal{O}_{K,w}\langle X \rangle$ and $a \in \mathcal{O}_{K,w}$ with $w(P(a)) > 0 = w(\frac{\partial}{\partial X_i} P(a))$ for some i, we can scale so that $P(X) \in \mathcal{O}_{K,v}$ and $a \in \mathcal{O}_{K,v}$. (Note: this move uses the fact that (K, v, D) has enough constants.) The hypothesis that $w(\frac{\partial}{\partial X_i} P(a)) = 0$, does not mean that $v(\frac{\partial}{\partial X_i} P(a)) = 0$. Rather, because w is a refinement of v, we can conclude from this and $w(P(a)) > 0$ that $v(P(a)) > 2v(\frac{\partial}{\partial X_i} P(a))$. By the strengthened version of DHL, there is some b such that $P(b) = 0$ and $v(a - b) = v(P(a)) - v(\frac{\partial}{\partial X_i} P(a))$. We have $w(a - b) = \tilde{w} \circ v(a - b) = \tilde{w} \circ v(P(a)) - \tilde{w} \circ v(\frac{\partial}{\partial X_i} P(a)) = w(P(a))$ as required. $\dashv$

With the above lemmata in place, we can reduce the our main problem to the case of pure characteristic zero valued D-fields.

Proposition 8.6 *Theorem 6.4 is equivalent to Theorem 6.3.*

Proof Using Test 7.4, the hypothesis of this proposition takes the form:

In any model of set theory in which GCH holds, if $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{vdf}}^{\mathrm{ac},0}$ is an expansion of the language $\mathcal{L}_{\mathrm{vdf}}^{\mathrm{ac},0}$ by predicates on the leading terms, K and L are saturated D-henselian fields of pure characteristic zero of the same cardinality considered as $\mathcal{L}$-structures, $\Gamma_K \equiv_{\mathcal{L}} \Gamma_L$, $\bigcup R_n(K) \equiv_{\mathcal{L}} \bigcup R_n(L)$, Γ_K and $\bigcup R_n(K)$ are $\mathcal{L}$-quantifier eliminable, and $f : A \to B$ is an isomorphism between small substructures of K and L, then f extends to an isomorphism between K and L.

Using the same test, the conclusion takes the form.

If $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{vdf}}^{\mathrm{ac},\omega}$ is a countable expansion of $\mathcal{L}_{\mathrm{vdf}}^{\mathrm{ac},\omega}$ by predicates on the leading terms, K and L are $\aleph_1$-saturated D-henselian fields of characteristic zero with $\mathcal{L}$-elementarily equivalent and quantifier eliminable value groups and residue fields, $A \subset K$ is a countable substructure, $a \in K$ is an element, and $f : A \hookrightarrow L$ is an $\mathcal{L}$-embedding, then f extends to an embedding of $A\langle\!\langle a \rangle\!\rangle$ into an elementary extension of L.

So we may assume the continuum hypothesis and take two saturated mixed characteristic valued D-field K and L of size $\aleph_1$ satisfying the above hypotheses for some langauge $\mathcal{L}$. We may (and do) assume that the map f actually induces an $\mathcal{L}$-isomorphism between $\bigcup_{n \in \omega} R_n(K)$ and $\bigcup_{n \in \omega} R_n(L)$ and between Γ_K and Γ_L

Let w_K be the valuation on K having valuation ring $\mathcal{O}_K[\frac{1}{p}]$ and w_L the valuation on L with valuation ring $\mathcal{O}_L[\frac{1}{p}]$. The structures (K, w_K, D) and (L, w_L, D) are valued D-fields of pure characteristic zero. Our task is to show that the hypotheses of Theorem 6.3 are true of these structures and that an embedding with respect to an expansion of $\mathcal{L}_{\mathrm{vdf}}^{\mathrm{lt},0}$ by predicates on w-zero leading terms induces an embedding of K into L for $\mathcal{L}_{\mathrm{vdf}}^{\mathrm{ac},\omega}$.

The saturation hypotheses on L and K imply that we may recover the w_K-residue field of K (respectively, the w_L-residue field of L) from the residue rings $R_n(K)$ and $R_n(L)$. That is, $\aleph_1$-saturation implies that the natural maps

$$\mathcal{O}_{K,v}/\mathfrak{m}_{K,w_K} \xrightarrow{\psi_K} \lim_{\substack{\longleftarrow \\ n \to \infty}} R_n(K)$$

and

$$\mathcal{O}_{L,v}/\mathfrak{m}_{L,w_L} \xrightarrow{\psi_L} \lim_{\substack{\longleftarrow \\ n \to \infty}} R_n(L)$$

are isomorphisms of multiplicative monoids. These maps preserve more than just the multiplicative structure. They are isomorphisms of $\mathcal{L}_{D-\mathrm{ring}}(\sigma, \sigma^{-1})$-structures as well: if $x, y \in K$ and $w_K(x+y) = w_K(x) = w_K(y)$, then for some $n \in \omega$ we have $v(x+y) < \min\{v(x) + nv(p), v(y) + nv(p)\}$ so that for $m > n$ the expression $\pi_{2m}(x) +_{2m,m} \pi_{2m}(y) = \pi_m(x+y)$ is well-defined. Likewise, the D and σ-structure on $K_{\mathfrak{m}_{K,w_K}}$ is determined by the structure on $\{K_n\}_{n\in\omega}$. Thus, the map f induces an isomorphism between the zero-leading term structure of K (with respect to w_K) and the zero-leading term structure of L (with respect to w_L).

We need to work to produce an angular component function for the coarsened valuation as the $\mathcal{L}^{\mathrm{ac},0}_{\mathrm{vdf}}$ structure is *not* canonically determined by the $\mathcal{L}^{\mathrm{ac},\omega}_{\mathrm{vdf}}$ structure. However, the indeterminacy may be traced to the choice of a section of $K^\times/\mathcal{O}^\times_{K,v} \to K^\times/\mathcal{O}_{K,v}[\frac{1}{p}]^\times$ so that we can keep it under control.

The angular component functions

$$K^\times \xrightarrow{\mathrm{ac}_n} (\mathcal{O}_K/p^{n+}\mathcal{O}_K)^\times$$

patch together to give a section

$$K^\times/1 + p^\infty\mathcal{O}_K \xrightarrow{\alpha} (\mathcal{O}_K/p^\infty\mathcal{O}_K)^\times$$

of the inclusion $(\mathcal{O}_K/p^\infty\mathcal{O}_K)^\times \hookrightarrow K^\times/1 + p^\infty\mathcal{O}_K$. However, α is *not* an angular component function. For α to be an angular component function we would need α restricted to $\mathcal{O}_K[\frac{1}{p}]^\times/1 + p^\infty\mathcal{O}_K$ to be the identity, but this is not the case as, for instance, $\alpha(p) = 1$.

The coarsened valuation w_K corresponds to the exact sequence

$$\begin{array}{ccccccccc} 1 & \to & \mathcal{O}_K[\frac{1}{p}]^\times/\mathcal{O}_K^\times & \to & K^\times/\mathcal{O}_K^\times & \to & K^\times/\mathcal{O}_K[\frac{1}{p}]^\times & \to & 1 \\ & & || & & || & & || & & \\ 1 & \to & \Gamma_{w|v} & \to & \Gamma_v & \to & \Gamma_w & \to & 1 \end{array}$$

Let $\psi : \Gamma_v \to \Gamma_{w|v}$ be a section of the inclusion $\Gamma_{w|v} \hookrightarrow \Gamma_v$.

From the splittings

$$K^\times/1 + p^n\mathcal{O}_K \xrightarrow{\mathrm{ac}_n} (\mathcal{O}_K/p^n\mathcal{O}_K)^\times$$

we obtain splittings

$$\Gamma_v \xrightarrow{\chi_n} K^\times/1 + p^n\mathcal{O}_K$$

of the sequences

$$1 \longrightarrow (\mathcal{O}_K/p^n\mathcal{O}_K)^\times \longrightarrow K^\times/1 + p^n\mathcal{O}_K \xrightarrow{v} \Gamma_v \longrightarrow 1$$

.

Let $\chi := \lim\limits_{\substack{\longleftarrow \\ n \to \infty}} \chi_n : \Gamma_v \to \lim\limits_{\substack{\longleftarrow \\ n \to \infty}} K^\times/1 + p^n\mathcal{O}_K = K_{\mathfrak{m}_{K,w}}$. Let $\beta : K^\times \to (\mathcal{O}_K[\frac{1}{p}]/p^\infty\mathcal{O}_K[\frac{1}{p}])^\times$ be defined by $\beta(x) := \alpha(x)\chi(\psi(v(x)))$. This function will serve as æ_0 for the coarsened valuation. We check now that it has the requisite properties.

First, we remark that β does take values in $(\mathcal{O}_K[\frac{1}{p}]/p^\infty\mathcal{O}_K[\frac{1}{p}])^\times$ as claimed. For any $x \in K$ we have $\psi(v(x)) \in \Gamma_{w|v}$ so that $\chi_n(\psi(v(x)) \in \mathcal{O}_K[\frac{1}{p}]^\times/1+p^n\mathcal{O}_K$. Thus, $\chi(\psi(v(x)) \in \mathcal{O}_K[\frac{1}{p}]^\times/1+p^\infty\mathcal{O}_K = \mathcal{O}_K[\frac{1}{p}]^\times/1+p^\infty\mathcal{O}_K[\frac{1}{p}] = (\mathcal{O}_K/p^\infty\mathcal{O}_K[\frac{1}{p}])^\times$. By construction, $\alpha(x) \in \mathcal{O}_K^\times/1+p^\infty\mathcal{O}_K \hookrightarrow (\mathcal{O}_K[\frac{1}{p}]/p^\infty\mathcal{O}_K[\frac{1}{p}])^\times$. Thus, $\beta(x) \in (\mathcal{O}_{K,w}/\mathfrak{m}_{K,w})^\times$.

Secondly, β is a homomorphism. This is clear from the construction.

Thirdly, β is a section. Let $x \in (\mathcal{O}_{K,w}/\mathfrak{m}_{K,w})^\times$. Write $x = \tilde{x}\chi(v(x))$. Note that $v(\tilde{x}) = 0$ so that $\tilde{x} \in \mathcal{O}_{K,v}^\times/1+\mathfrak{m}_{K,w}$. We compute

$$\begin{aligned}
\beta(x) &= \alpha(\tilde{x}\cdot\chi(v(x)))\cdot\chi(v(\tilde{x}\cdot\chi(v(x)))) \\
&= \alpha(\tilde{x})\cdot\alpha(\chi(v(x)))\cdot\chi(v(\tilde{x}))\cdot\chi(v(\chi(v(x)))) \\
&= \tilde{x}\cdot 1\cdot 1\cdot\chi(v(x)) \\
&= x
\end{aligned}$$

As noted in the section on leading terms, the fact that β is a section of $(\mathcal{O}_{K,w}/\mathfrak{m}_{K,w})^\times \hookrightarrow K_{\mathfrak{m}_{K,w}}$ implies already that β preserves addition as far as this makes sense.

Fourthly, β preserves the difference structure.

$$\begin{aligned}
\beta(\sigma(x)) &= \alpha(\sigma(x))\cdot\chi(\psi(v(\sigma(x)))) \\
&= \lim_{\overleftarrow{n\to\infty}} \text{æ}_n(\sigma(x))\cdot\lim_{\overleftarrow{n\to\infty}}\chi_n(\psi(v(x))) \\
&= \lim_{\overleftarrow{n\to\infty}} \sigma(\text{æ}_n(x))\cdot\lim_{\overleftarrow{n\to\infty}}\chi_n(\psi(v(x))) \\
&= \sigma(\alpha(x))\cdot\chi(\psi(v(x))) \\
&= \sigma(\alpha(x)\cdot\chi(\psi(v(x))) \\
&= \sigma(\beta(x))
\end{aligned}$$

The penultimate equality uses the fact that the image of χ is contained in the fixed field of σ. Why is this? An element x is equal to $\chi(\gamma)$ if and only if $v(x) = \gamma$ and $\alpha(x) = 1$. The element $\sigma(x)$ satisfies the same defining conditions.

Finally, β preserves the D-structure as far as this makes sense. The section χ also maps to D-constants as we have the functional equality $\alpha\circ\chi \equiv 1$. Thus, for any γ we have $w(D(\chi(\gamma))) > w(\chi(\gamma)) = \gamma$ for otherwise $\alpha(D(\chi(\gamma))) = D(\alpha(\chi(\gamma))) = D(1) = 0$ which is impossible.

So, let $x \in K^\times/1+\mathfrak{m}_{K,w}$. Let $\tilde{x} \in K^\times$ lift x. We suppose that $w(D\tilde{x}) = w(\tilde{x})$. This means that $v(D\tilde{x}) = v(\tilde{x}) + \gamma < v(\tilde{x}) + nv(p)$ for some $n \in \mathbb{Z}_+$. Write $x = \bar{x}\chi(v(x))$. Let $\tilde{\bar{x}} \in K^\times$ lift $\bar{x}$. Then as $w(Dy) > w(y)$ for any y lifting $\chi(v(x))$ we have $D\tilde{x} = (D\tilde{\bar{x}})\cdot\chi(v(x))(1+\mathfrak{m}_{K,w})$. We compute

$$\begin{aligned}
\beta(D\tilde{x}) &= \beta((D\tilde{\bar{x}})\chi(v(x))) \\
&= \beta(D\tilde{\bar{x}})\chi(\psi(v(\chi(v(x))))) \\
&= \alpha(D\tilde{\bar{x}})\chi(\psi(v(D\tilde{\bar{x}})))\chi(\psi(v(x))) \\
&= D\alpha(\bar{x})\chi(\psi(\gamma))\chi(\psi(v(x))) \\
&= D\alpha(\bar{x})\chi(\psi(\gamma+v(x)) \\
&= D(\alpha(\bar{x})\cdot\chi(\psi(v(D\tilde{x})))) \\
&= D(\beta(x))
\end{aligned}$$

In order to meet the hypotheses of Theorem 6.3, we replace L by a $|K|^+$-saturated elementary extension M of L. (Note that (L, w_L) is not even $\aleph_1$-saturated.) By Lemma 8.5, (L, w_L, D) is D-henselian. So, assuming Theorem 6.4 we obtain an embedding $g : (K, w_K, D) \to (M, w, D)$.

We use now the flexibility in the choice of $\mathcal{L}$. The w-residue field inherits a trace of the valuation v on K. That is, we may (and do) continue to consider $k_{K,w}$ has having the valuation v. From this (and w) we fully recover v. Thus the embedding $K \to M$ can be taken to preserve v as well!

□

9 The proofs in equicharacteristic zero

In the previous section we reduced the proof of the main theorem to the case of pure characteristic zero D-henselian fields. The proof we present in this section is a variant of the proof of the main theorem in [6]. While the statement *The proof of the main theorem in* [6] *goes through with only minor changes.* may be true, we present the details especially where the "minor changes" are not obvious.

Let us recall what needs to be proved. We are given two $\aleph_1$-saturated valued D-fields of pure characteristic zero M_1 and M_2 in some countable language $\mathcal{L}$ extending $\mathcal{L}^{\mathrm{ac},0}_{\mathrm{vdf}}$ with some not necessarily new predicates on the residue field and on the value group. We assume that $k_{M_1} \equiv k_{M_2}$, $\Gamma_{M_1} \equiv \Gamma_{M_2}$, and that the residue field and value group eliminate quantifiers in $\mathcal{L}$. We assume that $A \subseteq M_1$ is a countable substructure and that $f : A \hookrightarrow M_2$ is an $\mathcal{L}$-embedding and that $a \in M_1$. We need to show that f extends to an embedding of $A\langle\langle a\rangle\rangle$ into M_2.

Convention 9.1 *For the remainder of this section, "valued D-field" means "valued D-field of equicharacteristic zero with an angular component function of level* $\mathfrak{m}$*."*

Definition 9.2 If $P(X) = \sum p_\alpha \prod (D^j X)^{\alpha_j} \in K\langle X\rangle_D$ is a D-polynomial over the valued D-field K and $x \in K$ is an element with $v(x) = \gamma \in \Gamma_K$, then one *expects* $v(P(x)) = \min\{v(p_\alpha) + |\alpha|\gamma : \alpha \in \omega^{<\omega}\}$. We say that P *has the expected (or generic) valuation at* x if $v(P(x))$ is as expected.

The next lemma shows that if we control the valuation well enough, then we also control the angular component structure.

Lemma 9.3 *If K is a valued D-field, $P(X) \in K\langle X\rangle_D$ is a D-polynomial, $x \in K$ with $v(x) = \gamma \in \Gamma_K$, $v(D^j x) = \gamma$ for $j \leq \mathrm{ord} P$, and P has the expected valuation at x, then* $\text{æ}_0(P(x)) = \sum_{\{\alpha : v(p_\alpha)+|\alpha|\gamma = v(P(x))\}} \text{æ}_0(p_\alpha) \prod_{j=0}^\infty D^j \text{æ}_0(x)^{\alpha_j}$.

Proof Our hypothesis is that $v(D^j x) = \gamma$ for $j \leq \mathrm{ord} P$ implies that $v(p_\alpha \prod (D^j x)^{\alpha_j}) = v(p_\alpha) + |\alpha|\gamma$ for each multi-index α. Let $T := \{\alpha : v(p_\alpha) + |\alpha|\gamma = v(P(x))\}$. By the ultrametric triangle inequality, we have $v(P(x)) = v(\sum_{\alpha \in T} p_\alpha \prod (D^j X)^{\alpha_j})$. Thus, by Lemma 3.3 we have $\text{æ}_0(P(x)) = \sum_{\alpha \in T} \text{æ}_0(p_\alpha \prod (D^j X)^{\alpha_j}) = \sum_{\alpha \in T} \text{æ}_0(p_\alpha) \prod (D^j(\text{æ}_0(x))^{\alpha_j}$ as claimed. □

Lemma 9.4 *Let K be a valued D-field considered as an $\mathcal{L}$-structure with L and L' two immediate extensions. If $L \cong_{\mathcal{L}_{\mathrm{vdf}K}} L'$, then $L \cong_{\mathcal{L}_K} L'$.*

Proof Let $g : L \to L'$ be an $\mathcal{L}_{\mathrm{vdf}K}$-isomorphism. Then g induces the identity map on $L_0 = K_0 = L'_0$. As the new functions and relations in $\mathcal{L}_K$ are all defined on the leading terms exclusively, the map g respects them as well. □

We need to pin down the structure on the henselization of a valued D-field in terms of the structure on the original field.

Lemma 9.5 *If K is a valued D-field considered as an $\mathcal{L}$-structure, then the henselization K^h of K (or more accurately, the field of fractions of the henselization, $\mathcal{O}_K^h$, of the ring of integers of K) has a unique structure of a valued D-field in the language $\mathcal{L}_K$.*

Proof Lemma 7.11 of [6] shows that when $v(e) > 0$, there is a unique $\mathcal{L}_{\mathrm{vdf}\,K}$-structure on K^h. In the case that $v(e) = 0$, D is interdefinable with σ (and with σ^{-1}). By the universal property of the henselization $\sigma : \mathcal{O}_K \to \mathcal{O}_K \hookrightarrow \mathcal{O}_K^h$ induces a unique map $\sigma : \mathcal{O}_K^h \to \mathcal{O}_K^h$ as does σ^{-1}. Hence, in this case as well there is a unique $\mathcal{L}_{\mathrm{vdf}\,K}$-structure on K^h.

As a general rule $K_0 = (K^h)_0$. Thus, by Lemma 9.4 the $\mathcal{L}_K$-structure on K^h is also determined. □

The next lemma concerns the structure of an extension obtained by adjoining a new element to the residue field. The proof of the corresponding lemma in [6] (Lemma 7.12) used the hypothesis that $v(e) > 0$ substantially, though the use is removable. The proof given below recasts that proof with this extraneous hypothesis removed.

Lemma 9.6 *If K is a valued D-field considered as an $\mathcal{L}$-structure, $p(x) \in S_{1,k}(k_K)$ is a one-type in the residue field sort, $P(X) \in \mathcal{O}_K\langle X\rangle_D$ is a D-polynomial over K for which*

- *$p(x) \vdash \pi(P)(x) = 0$,*
- *P and $\pi(P)$ have the same total degree, and*
- *if $Q(X) \in \mathcal{O}_K/\mathfrak{m}_K\langle X\rangle$ with $p(x) \vdash Q(x) = 0$, then $\pi(P) \ll Q$;*

then there is a unique $\mathcal{L}_K$-structure on $L = K\langle\!\langle b\rangle\!\rangle$ for which $P(b) = 0$, $v(b) \geq 0$, and $\pi(b) \models p$.

Proof Lemma 7.12 of [6] shows the existence and uniqueness of the $\mathcal{L}_{\mathrm{vdf}\,K}$-structure on L in the case of $v(e) > 0$. Moreover, every element c of L is of the form $\frac{R(b)}{Q(b)}$ with $R, Q \ll P$. It follows that $R(X)$ and $Q(X)$ have the expected valuation at b, and, therefore, by Lemma 9.3 $\mathrm{ac}_0(c)$ may be computed from $\mathrm{ac}_0(b)$. As there is no extension of the valuation group, this observation shows that the $\mathcal{L}_K$-structure is pinned down.

In the case of $v(e) = 0$, we need a different argument. We prove uniqueness first and then show existence. In the course of showing that the $\mathcal{L}_{\mathrm{vdf}\,K}$-structure is uniquely determined, we will show that every new element of the residue field is a $\mathcal{O}_K/\mathfrak{m}_K$-rational function of $\mathrm{ac}_0(b)$ and its conjugates under integral powers of σ. Thus, included in the proof of the uniqueness of the $\mathcal{L}_{\mathrm{vdf}\,K}$-structure is a proof of the uniqueness of the $\mathcal{L}_K$-structure.

As in the proof of Lemma 9.5, it suffices to work with σ and σ^{-1} polynomials rather than D-polynomials. Substituting $e^{-1}(\sigma - \mathrm{id})$ for the operator D and abusing notation, we regard P as a σ-polynomial.

Let $d := \mathrm{ord} P$. Let $K_n := K(b, \ldots, \sigma^n(b))$ for $n \geq 0$, $K_{-1} := K$, and $L_+ := \bigcup_{n\in\omega} K_n$.

We show now that the $\mathcal{L}_{\mathrm{vdf}\,K}$-structure on K_n is uniquely determined. When $n \leq d$, then every element of K_n may be expressed as quotient of σ-polynomials simpler than P. Just as in the case of $v(e) > 0$, the $\mathcal{L}_{\mathrm{vdf}_K}$-structure is determined on K_n.

The case of $n = d + j$ with $j \geq 0$ requires some work. Write $P(X) = F(X, \sigma(X), \ldots, \sigma^d(X))$ with $F(X_0, \ldots, X_d) \in \mathcal{O}_K[X_0, \ldots, X_d]$. Set $G(Y) := F(b, \ldots, \sigma^{d-1}(b), Y)$. Then $G(Y)$ is a minimal polynomial of $\sigma^d(b)$ over K_{d-1}. By hypothesis, $\pi(b)$ is a simple root of $\pi(G)(Y)$. As σ induces an automorphism of the residue field, $\sigma^j(b)$ is a simple root of $\pi(\sigma^j(G))(Y)$ for each $j \geq 0$. Thus, K_{d+j} is contained in the strict henselization of K_d for each $j \geq 0$. Moreover, for each $j \geq 0$, as a valued field the extension K_{d+j}/K_{d+j-1} is characterized over K_{d+j-1} and $k_K\langle\langle \pi(b)\rangle\rangle$ by $\sigma^j(b)$ is the unique solution to $\sigma^j(G)(Y) = 0$ and $\pi(Y) = \sigma^j(\pi(b))$.

To get all of L we need to consider $\bigcup_{n=0}^{\infty} \sigma^{-1}(L_+)$. We do this by working with $H(Y) := F(Y, \sigma(b), \ldots, \sigma^d(b))$ instead of $G(Y)$.

For existence, pick some realization $\bar{b} \models p$. Give $K' := K(X_0, \ldots, X_{d-1})$ the structure of a valued field with residue field contained in $\mathcal{O}_K/\mathfrak{m}_K\langle\langle \bar{b}\rangle\rangle$ by defining $v(\sum p_\alpha X^\alpha) := \min\{v(p_\alpha)\}$ and $\pi(X_i) := \sigma^i(\bar{b})$. It is a routine matter to check that this defines a valuation.

Let L be the strict henselization of K'. Let $y \in \mathcal{O}_L$ with $F(X_0, \ldots, X_{d-1}, y) = 0$ and $\pi(y) = \sigma^d(\bar{b})$. Note that Hensel's Lemma guarantees the existence of y.

Define $\tilde{\sigma} : K' \to L$ by $\tilde{\sigma}|_K := \sigma$, $\tilde{\sigma}(X_i) := X_{i+1}$ for $0 \leq i < d-1$ and $\tilde{\sigma}(X_{d-1}) := y$. By the irreducibility of P, $X_1, \ldots, X_{d-1}, y$ are algebraically independent over K. Hence, a unique field homomorphism is specified by the above conditions. Using the fact that $v(y) > 0$ and σ preserves the valuation on K, we see that the image of $\tilde{\sigma}$ on $\mathcal{O}_{K'}$ is contained in $\mathcal{O}_L$. Thus, by the universality property of the strict henselization, there is an extension of $\tilde{\sigma}$ to $\mathcal{O}_L$. The extension need not be unique, but on the (inversive) difference field generated by X_0 over K, $K\langle\langle X_0\rangle\rangle$, it is unique as it is the only such map lifting σ. Likewise, we find an map $\tilde{\sigma}^{-1} : \mathcal{O}_L \to \mathcal{O}_L$ which when restricted to $\mathcal{O}_{K\langle\langle X_0\rangle\rangle}$ is an inverse to $\tilde{\sigma}$. As both $\tilde{\sigma}$ and $\tilde{\sigma}^{-1}$ preserve the valuation ring, they must preserve the valuation itself. □

As remarked in the introduction, the main reason for introducing angular component functions is to deal with radical extensions. We do this with the next two lemmata.

Lemma 9.7 *If K is a valued D-field and $\pi : \mathcal{O}_K \to k_K$ is surjective, then for any $b \in K^\times$ there is some $\epsilon \in K$ with $v(\epsilon) = v(b)$ and $\text{æ}_0(\epsilon) = 1$ (and, therefore, $v(D\epsilon) > v(\epsilon)$).*

Proof As π is surjective, there is some $c \in \mathcal{O}_K^\times$ with $\pi(c) = \text{æ}_0(b)^{-1}$. Set $\epsilon := c \cdot b$. □

Lemma 9.8 *If K is a valued D-field and $\eta \in K^\times$ with $\text{æ}_0(\eta) = 1$ and $\mathrm{tp}(v(\eta)) \vdash n|x$ but $v(\eta) \notin mvK$ for any $m > 1$ with $m|n$, then the field $K\langle\langle \epsilon\rangle\rangle$ where $\epsilon^n = \eta$ has a unique $\mathcal{L}_K$-structure with $\text{æ}_0(\epsilon) = 1$.*

Proof That the valuation structure on $K(\epsilon)$ is determined is well-known. Moreover, since ϵ has the expected valuation for every polynomial over K of degree less than n, the angular component structure is pinned down. There is no residue extension, so we need not worry about the possible new $\mathcal{L}_K$-relations. The new elements of the value group all lie in the group generated by vK and $v(\epsilon)$ which is definable from $v(\eta)$.

In the case that $e = 0$, there is nothing more to do as in this case $D\epsilon \in K(\epsilon)$. We assume now that $e \neq 0$.

As σ must preserve the valuation, we see that $\sigma(\epsilon) = \omega \cdot \epsilon$ where $v(\omega) = 0$. As σ commutes with æ_0, we see that $\pi(\omega) = \text{æ}_0(\omega) = 1$. The image of ϵ under σ must be a root to $X^n = \sigma(\eta)$. Hence, ω is a root to $X^n = \frac{\sigma\eta}{\eta}$. Note that $\pi(\frac{\sigma(\eta)}{\eta}) = 1$. As the residue characteristic is zero, there is a unique element $\tilde{\omega}$ of the henselization of $K(\epsilon)$ with $\pi(\tilde{\omega}) = 1$ and $\tilde{\omega}^n = \frac{\sigma(\eta)}{\eta}$. Thus, by the universal property of the henselization, there is a unique embedding of $K\langle\langle\epsilon\rangle\rangle$ into its henselization compatible with $\sigma|_K$, $\sigma(\epsilon)^n = \sigma(\eta)$, and $\text{æ}_0(\sigma(\epsilon)) = 1$. $\square$

The next lemma is the analogue of the above lemma for transcendental valuations.

Lemma 9.9 *If K is a valued D-field, $p(x) \in S_{1,\Gamma}(vK)$, and $p(x) \vdash \{nx \neq v(b) : n \in \mathbb{Z}_+, b \in A\}$, then $K(\epsilon)$ has a unique $\mathcal{L}_K$-structure with $D\epsilon = 0$, $\text{æ}_0(\epsilon) = 1$, and $v(\epsilon) \models p$.*

Proof Lemma 7.8 of [6] shows that $K(\epsilon)$ has a unique $\mathcal{L}_{\mathrm{vdf}\,K}$-structure with $v(\epsilon) modelsp$ and $D\epsilon = 0$. As there is no extension of the residue field and p determines the extension on the value group, this structure is also determined. As every polynomial over K has the generic valuation at ϵ, the expansion to $\mathcal{L}^{\mathrm{ac},0}_{\mathrm{vdf}\,K}$ is also determined. $\square$

We proceed now to prove Theorem 6.4 (and, hence, Theorem 6.3).

Proof of Main Theorem:

We follow the strategy and arguments of [6] quoting the above lemmata in some places to guarantee that the arguments work in $\mathcal{L}$ with $v(e) \geq 0$. We extend the embedding $f : A \hookrightarrow M_2$ in stages.

We find a countable elementary submodel $N_1 \prec M_1$ with $A\langle\langle a\rangle\rangle \subseteq N_1$.

We extend f so that $\pi(\mathcal{O}_A) = k_{N_1}$ using Lemma 9.6. That is, if $\bar{b} \in k_{N_1} \setminus \pi(\mathcal{O}_A)$, then we choose $P(X) \in \mathcal{O}_A\langle X\rangle_D$ so that T.deg$\pi(P) =$ T.deg(P) and $\pi(P)$ is a minimal D-polynomial for b over $\mathcal{O}_A/\mathfrak{m}_A$. As we have assumed quantifier elimination for the residue field, we may extend f to b. As N_1 is D-henselian, there is some $\tilde{b} \in \mathcal{O}_{N_1}$ with $\pi(\tilde{b}) = \bar{b}$. Likewise, there is some $\tilde{c} \in \mathcal{O}_{M_2}$ with $\pi(\tilde{c}) = f(\bar{b})$.

As the residue characteristic is zero, DHL applies to P at $\tilde{b}$, and, hence, also to $f(P)$ at $\tilde{c}$. Let $b \in \mathcal{O}_{N_1}$ with $P(b) = 0$ and $\pi(b) = \bar{b}$ and $c \in \mathcal{O}_{M_2}$ with $f(P)(c) = 0$ and $\pi(c) = f(\bar{b})$. By Lemma 9.6 there is an extension of f to $A\langle\langle b\rangle\rangle$ determined by $b \mapsto c$.

We then extend f so that $vA = \Gamma_{N_1}$. There are two different steps involved with this kind extension.

If $\gamma \in \Gamma_{N_1}$ but $n\gamma \notin vA$ for all $n \in \mathbb{Z}_+$, then Lemma 9.9 $\gamma = v(b)$ for some b in the domain.

In the other case, take n minimal with $n\gamma \in vA$. If $n = 1$, there is nothing to do. Otherwise, since $\mathcal{O}_A/\mathfrak{m}_A = k_{N_1}$ and π is surjective on $\mathcal{O}_{N_1}$, by Lemma 9.7 there is some $\eta \in A$ with $v(\eta) = n\gamma$ and $\text{æ}_0(\eta) = 1$. Let $\tilde{\epsilon} \in N_1$ with $v(\tilde{\epsilon}) = \gamma$ and $\text{æ}_0(\tilde{\epsilon}) = 1$. Applying Hensel's Lemma to $X^n - \frac{\eta}{\tilde{\epsilon}^n}$ at 1 we find that there is some $\epsilon \in N_1$ with $\epsilon^n = \eta$ and $\text{æ}_0(\epsilon) = 1$. Likewise, there is some $\zeta \in M_2$ with $\zeta^n = f(\eta)$ and $\text{æ}_0(\zeta) = 1$. By Lemma 9.8 we may extend f to $A\langle\langle\epsilon\rangle\rangle$ by sending $\epsilon \mapsto \zeta$.

We have reduced to the case that N_1 is an immediate extension of A. If one is willing to carefully examine the proofs in section 7.2 of [6], then one sees that the hypothesis that A has enough constants is not used when dealing with algebraic extensions or when dealing with extensions for which $D(b)$ is rational over b. This shows that it is possible to extend the embedding inside N_1 so that A has enough constants. However, as I have disavowed such tests of the readers' patience, we must follow a different route.

We will find a countable unramified extension $A^{(2)}$ of A on which an extension of f is defined and which has enough constants. We then take $N_1 \preceq N_1^{(2)} \prec M_1$ a countable model with $A^{(2)} \subseteq N_1^{(2)}$. We repeat the above arguments extending so that $N_1^{(2)}$ is an immediate extension of the domain of f. We then find $A^{(3)}$, a countable unramified extension of $A^{(2)}$ on which an extension of f is defined and which has enough constants, and so on. Eventually, we let N be the direct limit of the $N_1^{(i)}$'s and we will have that N is a countable elementary submodel of M_1, is an immediate extension of the domain of f, and $a \in N$.

To extend f so that the domain has enough constants, take $\gamma \in \Gamma_A$. As M_1 is D-henselian and $\aleph_1$-saturated, we can find $\epsilon \in M_1$ with $D\epsilon = 0$, $v(\epsilon) = \gamma$, and $\text{æ}_0(\epsilon) \notin k_A^{alg}$. One computes easily that every polynomial over A has the expected valuation at ϵ so that the $\mathcal{L}_A$-structure on the unramified extension $A\langle\langle\epsilon\rangle\rangle$ is determined by $p(x) := \text{tp}(\text{æ}_0(\epsilon)/k_A)$. Using the fact that M_2 is an $\aleph_1$-saturated D-henselian field we find $\zeta \in M_2$ with $v(\zeta) \models f(\text{tp}(\gamma/\Gamma_A))$, $D\zeta = 0$, and $\text{æ}_0(\zeta) \models f(p)$.

From now on, we may assume that A has enough constants and that N is an immediate extension of A. Given Lemma 9.4, the proof for this last step is (almost) the same as the proof in [6]. The sole use of the hypothesis $v(e) > 0$ in the original proof is cosmetic: in the last paragraph of Lemma 7.47 the fact that $\mathcal{D}_e(R)$ is henselian is used. However, when $v(e) = 0$, this ring is isomorphic to $R \times R$ so that one should apply the universal property of the Cartesian product and of the henselization to find the unique extension ψ.

Proposition 7.51 of [6] shows that there is an extension of f to an $\mathcal{L}_{\text{vdf}K}$-embedding of N into M_2. By Lemma 9.4, this is also an $\mathcal{L}_K$-embedding.

✠

10 Eliminating the leading terms

The angular component functions and leading term structures used in this paper clutter our results. In this section we note that the results of the previous section suffice for a complete axiomatization of the theory of a D-henselian field of characteristic zero.

To begin, we note Theorem 6.3 implies a complete axiomatization in $\mathcal{L}_{\text{vdf}}$.

Theorem 10.1 *Let K be a D-henselian field of characteristic zero. Then the theory of K in $\mathcal{L}_{\text{vdf}}$ is determined by the theory of D-henselian fields of characteristic zero, the atomic theory of K, the theory of the value group and the theory of $\bigcup_{n\in\omega} \mathcal{O}_K/p^{n+}\mathcal{O}_K$ considered as a many sorted structure.*

Proof: As the statement of this theorem is absolute, we may and do assume GCH. Let $K' \succ K$ be a saturated elementary extension of K'. Let $\chi : vK' \to ((K')^D)^\times$ be a section of the valuation on K'. If the residue characteristic of K is p, define

$\text{æ}_n(x) := \frac{x}{\chi(v(x))} + p^{n+}\mathcal{O}_{K'}$ for $x \in (K')^\times$ and $n \in \omega$. If the residue characteristic is zero, set $\text{æ}_0(x) := \frac{x}{\chi(v(x))} + \mathfrak{m}_{K'}$ for $x \in (K')^\times$. With these functions K' is a D-henselian field of characteristic zero with a system of angular component functions. By Theorem 6.3, the theory of K' in $\mathcal{L}_{\text{vdf}}^{\text{ac},\omega}$ is determined by this fact and the theory of its value group and of $\bigcup R_n(K')$. Hence, the theory of its reduct to $\mathcal{L}_{\text{vdf}}$ is determined by the same things and thus, the same is true of K. ⊣

If we specialize K, then the statement of Theorem 10.1 becomes cleaner.

Corollary 10.2 *The completions of the theory of D-henselian fields of pure characteristic zero are determined by specifying the atomic theory, the theory of the residue field, and the theory of the value group.*

Remark 10.3 The generalized power series construction of [6] shows that we may take any D-linearly closed field of characteristic zero and choose for e any element of non-negative valuation in the generalized power series ring.

Corollary 10.4 *Let p be a rational prime. Let k be a field of characteristic p having no degree p extensions. Let $K := W_{p^\infty}(\mathbb{F}_p^{alg})[\frac{1}{p}]$ be the field of fractions of the Witt vectors over $\mathbb{F}_p^{alg}$. Let $\sigma : K \to K$ be the relative Frobenius. Let $v : K \to \mathbb{Z} \cup \{\infty\}$ be the p-adic valuation on K. Then the theory of (K, v, σ) is axiomatized by saying that the function $D(x) := \sigma(x) - x$ makes K into a D-henselian field of characteristic zero with $e = 1$, residue field satisfies $\mathrm{Th}(k)$, the value group is a Z-group with least positive element $v(p)$, and $D(x) \equiv x^p - x \pmod{p}$ for $x \in \mathcal{O}_K$. Moreover, we have quantifier elimination in $\mathcal{L}_{\text{vdf}}(\{\text{æ}_n\}_{n\in\omega})$ relative to the residue field. In particular, when $k = k^{alg}$, we have absolute quantifier elimination.*

Proof The rings $R_n(K)$ are definably isomorphic to $W_{p^n}(k)$ which are themselves bi-interpretable with k. □

The reader may substitute finitely ramified extensions of $W_{p^\infty}(k)[\frac{1}{p}]$ where k is an algebraically closed field of characteristic p and powers of the relative Frobenius in the above corollary to obtain other axiomatizations of concrete theories of standard valued difference fields.

Remark 10.5 The $\mathcal{L}_{\text{vdf}}$-structure on $W_{p^\infty}(\mathbb{F}_p^{alg})[\frac{1}{p}]$ is an expansion of the Teichmüller structure considered in [8] as the Teichmüller representives are defined by the equation $\sigma(x) = x^p$.

Remark 10.6 There are D-henselian fields of mixed characteristic in which the residue field is differential field. Since we must require the residue field to be D-linearly closed, the residue fields necessarily have infinite degree of imperfection.

References

[1] J. Ax and S. Kochen, *Diophantine problems over local fields. III. Decidable fields*, Ann. of Math. (2) **83** (1966), 437–456.
[2] Ş. Basarab and F.-V. Kuhlmann, *An isomorphism theorem for Henselian algebraic extensions of valued fields*, Man. Math. **77** (1992), no. 2-3, 113–126.
[3] L. Bélair, *Types dans les corps valus munis d'applications coefficients*, Illinois J. Math. **43** (1999), no. 2, 410–425.
[4] L. Bélair, A. Macintyre, and T. Scanlon, *The relative Frobenius*, in preparation.
[5] A. Macintyre, *On definable subsets of p-adic fields*, J. Sym. Logic **41** (1976), no. 3, 605–610.
[6] T. Scanlon, *A model complete theory of valued D-fields*, J. Symbolic Logic (to appear).

[7] T. Scanlon, *Model theory of valued D-fields*, Ph. D. thesis, Harvard University, May 1997.
[8] L. van den Dries, *On the elementary theory of rings of Witt vectors with a multiplicative set of representatives for the residue field*, Man. Math. **98** (1999), no. 2, 133–137.