Lascar and Morley ranks differ in differentially closed fields

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We note here, in answer to a question of Poizat, that the Morley and Lascar ranks need not coincide in differentially closed fields. We will approach this through the (perhaps) more fundamental issue of the variation of Morley rank in families. We will be interested here only in sets of finite Morley rank. § 1 consists of some general lemmas relating the above issues. § 2 points out a family of sets of finite Morley rank, whose Morley rank exhibits discontinuous upward jumps. To make the base of the family itself have finite Morley rank, we use a theorem of Buium.
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1 Definability of Morley rank

We will say that Morley rank is *definable* (respectively *upward*, resp. *downward semi-definable*) if for every set of parameters A and A-definable family of definable sets $E_b (b \in B)$ and $b \in B$, there is an A-definable set $B' \subseteq B$ such that $b \in B'$ and $MR(E_{b'}) = MR(E_b)$ (resp. $\geq$, $\leq$) for $b' \in B'$.

Lemma 1.1 *Let T be a theory of finite Morley rank. If Morley and Lascar ranks coincide on definable sets, then Morley rank is downward semi-definable.*

Proof Suppose Morley rank is not downward semi-definable, and let E_b $(b \in B)$ be an A-definable family demonstrating this. That is, there is some $b^* \in B$ so that for every A-definable set $B' \subseteq B$ with $b \in B'$ there is some $b' \in B'$ with $MR(E_{b'}) > MR(E_{b^*})$. Replace B with an A-definable set of minimal Morley rank and degree containing b; so that now $MR(\mathrm{tp(b^*/A)}) = \mathrm{MR(B)} := \mathrm{m}$. Let $d := MR(E_{b^*})$. For $b \in B$, let $E'_b := {E_b}^m$. Then $MR(E'_b) = mMR(E_b)$. So it is always a multiple of m. We have $MR(E'_{b^*}) = md$, while for many b', $MR(E'_{b'}) \geq md + m$.
For $B' \subset B$, let

$$X_{B'} := \{(e, b) : b \in B', e \in E'_b\}$$

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and $X := X_B$. Replace B with some A-definable $B' \subseteq B$ with $b^* \in B'$ and the property that for every A-definable $B'' \subseteq B'$ with $b^* \in B''$, $(MR(X_{B''}), dM(X_{B''})) = (MR(X_{B'}), dM(X_{B'}))$.

If $(e, b) \in X$ and b is not generic in B, then $b \in B'$ for some A-definable subset of B with $MR(B') < MR(B)$ and $(e, b) \in X_{B'}$. Since $X_{B \setminus B'}$ and X_B have the same Morley rank and degree by the above reduction, $MR(X'_B) < MR(X)$; so, $MR(\mathrm{tp(e, b/A)}) < \mathrm{MR(X)}$.

On the other hand, if $(e, b) \in X$, and $b \in B$ is generic, then $U(\mathrm{tp(e, b/A)}) \leq \mathrm{md + m}$. But $MR(E'_b) \geq md + m$ for infinitely many $b \in B$; so $MR(X) > md + m$. Thus $U(\mathrm{tp(e, b/A)}) < \mathrm{MR(X)}$ for any $(e, b) \in X$. So $U(X) < MR(X)$, a contradiction.

2 A non-definable family

We now work with differential fields of characteristic 0, and fix a universal domain $\mathbb{U}$ (a saturated differentially closed field.)

Our plan is to produce a finite rank definable family of abelian varieties whose Manin kernels exhibit non-definable jumps in Morley rank. One difficulty is that there does not exist a definable family of abelian varieties containing a copy of every abelian variety of a given dimension. However, there are definable families of abelian varieties containing isomorphic copies of every *principally polarized* abelian variety of a given dimension.

For every abelian variety A there is another abelian variety $\breve{A}$, called the dual abelian variety, which parametrizes the line bundles on A algebraically equivalent to zero. A polarization is an isogeny $\lambda : A \to \breve{A}$. A polarization is principal if it is an isomorphism. A *principally polarized abelian variety* is an abelian variety A given together with a principal polarization $\lambda : A \to \breve{A}$. Not all abelian varieties admit a principal polarization, but elliptic curves always do.

Theorem 2.1 ([6] VII §2) *Let L be an algebraically closed field, and g a positive integer. There exists a definable family $\{(A_b, \lambda_b) : b \in F\}$ of g-dimensional principally polarized abelian varieties, such that every principally polarized g-dimensional abelian variety over L is isomorphic to some $(A_\alpha, \lambda_\alpha)$.*

Let E_t be an elliptic curve with j-invariant t. Given t, t', let $E(t, t') := E_t \times E_{t'}$. Given also an integer n, there exist group-theoretic isomorphisms ι between the finite n-torsion subgroups of E_t and of $E_{t'}$. Let $E(t, t', \iota, n)$ be the quotient of $E_t \times E_{t'}$ by the graph of ι. Then $A := E(t, t', \iota, n)$ is an Abelian variety of dimension 2. When $E_t, E_{t'}$ are not isogenous, A has precisely two connected definable subgroups of Morley rank 1, namely the images of E_t and of $E_{t'}$. Their intersection has order n^2. For a general choice of ι, $E(t, t', \iota, n)$ need not admit a principal polarization. However, if we choose ι to be anti-symplectic (ie $\langle \iota(x), \iota(y) \rangle_{E_{t'}} = \langle y, x \rangle_{E_t}$ where $\langle \cdot, \cdot \rangle$ is the Weil pairing (See [4] §16 for the general theory of the Weil pairing) then $E(t, t', \iota, n)$ is self-dual.

Lemma 2.2 *If $\iota : E_t[n] \to E_{t'}[n]$ is an anti-symplectic isomorphism of the n-torsion points, then $A := E(t, t', \iota, n)$ has a natural principal polarization.*

Proof Since ι is an anti-symplectic map, the graph of ι is isotropic for the pairing on $E_t \times E_{t'}$:

$$\begin{aligned}\langle (x,\iota(x)),(y,\iota(y))\rangle_{E_t\times E_{t'}} &= \langle x,y\rangle_{E_t}\cdot\langle \iota(x),\iota(y)\rangle_{E_{t'}}\\ &= \langle x,y\rangle_{E_t}\langle y,x\rangle_{E_t}\\ &= 1\end{aligned}$$

Since $\#(E_t \times E_{t'})[n] = n^4$ and the pairing is perfect, a maximal isotropic space has size n^2 which is the size of the graph of ι. Hence, the graph of ι is a maximal isotropic subspace.
The lemma now follows from the more general lemma:

Lemma 2.3 *Let A be a principally polarized abelian variety identified with its dual via the polarization. Let $\Gamma \subseteq A[n]$ be a maximal isotropic subgroup of the n-torsion subgroup of A. Let $B = A/\Gamma$. Then B also admits a principal polarization.*

Proof Let $\pi : A \to B$ be the quotient map. Let $\phi : B \to A$ be defined by $b \mapsto [n]a$ where a is any choice of a pre-image of b under π. ϕ induces a dual exact sequence

$$0 \longrightarrow \mathrm{Hom}(\ker\phi, \mu_n) \longrightarrow A \longrightarrow \check{B} \longrightarrow 0$$

where μ_n is the group of n-th roots of unity (See [5] III §15). The kernel of ϕ is $A[n]/\Gamma$. Since Γ is isotropic, the pairing on $A[n]$ descends to a pairing $(A[n]/\Gamma) \times \Gamma \to \mu_n$. Since the pairing is perfect and $\#A[n] = (\#\Gamma)^2$, via this pairing $\Gamma = \mathrm{Hom}(A[n]/\Gamma, \mu_n)$.
Thus, the above exact sequence is

$$0 \longrightarrow \Gamma \longrightarrow A \longrightarrow \check{B} \longrightarrow 0$$

That is, $\check{B}$ is the quotient $A/\Gamma = B$.

Lemma 2.4 *Let F' be a Zariski (resp. Kolchin) closed subset of F, the definable parameter space for two dimensional principally polarized abelian varieties of Theorem 2.1. Assume F' has a Zariski (resp. Kolchin) dense subset $\{t_1, t_2, \ldots\}$, such that A_{t_n} is isomorphic to some $E(t, t', \iota, n)$ with ι anti-symplectic as above. Then for generic $t \in F'$, A_t is a simple abelian variety.*

Proof Otherwise, a generic A_t contains two elliptic curves. Their intersection is necessarily finite, say of order m. But then infinitely many A_{t_n} must contain two elliptic curves with intersection of order m. For $n > m$, this contradicts the remarks above.
At this point it is quite easy to see that exists in DCF_0 a definable family of definable sets, whose generic element is strongly minimal, but with densely many sets of Morley rank 2. Thus:

Corollary 2.5 *In DCF_0, Morley rank is not downwards semi-definable.*

However, since DCF_0 does not have finite Morley rank, lemma 1.1 does not directly apply. At this point we quote a theorem from [1].

Theorem 2.6 (Buium [1]) *Let (A, λ) be any principally polarized abelian variety of maximal δ-rank. There exists a definable family $\{(A_t, \lambda_t) : t \in F_1\}$, containing a definably isomorphic copy of every principally polarized abelian variety isogenous to A, and such that F_1 has finite Morley rank.*

We leave the notion of *δ-rank* undefined here since we need only the facts that:

- A generic elliptic curve has maximal δ-rank.
- The property of having maximal δ-rank is isogeny invariant.
- The product of two abelian varieties each of maximal δ-rank is also of maximal δ-rank.

It seems likely that the δ-rank condition is unnecessary in Buium's theorem, but we leave this issue aside.

Corollary 2.7 *There exists a finite Morley rank definable subset Y, such that Morley rank is not downwards semi-definable.*

Proof Pick t, t' algebraically independent over k, the field of differential constants of $\mathbb{U}$. Let $J_t, J_{t'}$ be elliptic curves with j-invariants t, t'. Let $A := J_t \times J_{t'}$, and let F_1 be a family as guaranteed to exist by Theorem 2.6. Given n, pick $c = c(n) \in F_1$ with A_c isomorphic to $E(t, t', \iota, n)$. Let F_2 be the Kolchin closure of the set $\{c(1), c(2), ...\}$. Let b be a generic element of F_2. By Lemma 2.4, A_b is a simple Abelian variety. If A_b were isogenous to an Abelian variety defined over k, this would be guaranteed by a certain formula true of b, and the same formula would hold of infinitely many $c(n)$; hence A would also have this property, contradicting the choice of t, t'. Thus A_b is a simple, non-isotrivial Abelian variety.
For $t \in F_2$, let M_t be the Manin kernel of A_t. M_t is uniformly definable over t (cf. [2]). Then (cf. [3]) M_t has Morley rank 1 for generic $t \in F'$ (when A_t is a nonisotrivial simple Abelian variety.) But it has Morley rank 2 for each $t = c(n)$ (when A_t is isogenous to a product of elliptic curves.) Thus Morley rank is not downward semi-definable in $Y = \{(a, t) : t \in F^{\#}, a \in M_t\}$.

Corollary 2.8 *Morley and Lascar rank do not agree on definable sets in DCF_0.*

Proof Since Y has finite Morley rank, with the structure induced from the ambient differentially closed field, Lemma 1.1 applies.

Question 2.9

Marker and Pillay have noted that on 0-definable sets of differential order 2, Lascar and Morley ranks are the same. Examples similar to the one produced above have order at least 5. Is there a theorem responsible for this gap?

References

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