Motivic Integration: An outsider's tutorial

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What is motivic integration? (An etymological answer)

The theory of motives was introduced by Grothendieck as a way to explain cohomological arguments in algebraic geometry (and especially such theorems as independence of the results from the specific cohomology theory used) geometrically.

Integration, at first at least, really meant integration in the Lebesgue sense. Motivic integration interpolates p-adic integration. In the more recent developments, the integrals have a more formal character along the lines of the integrals in de Rham cohomology rather than those defined measure theoretically.

What is motivic integration? (Historical answer)

- Motivic integration as introduced by Kontsevich in 1995 is a theory which makes sense of some integrals over the $\mathbb{C}[[t]]$-valued points of algebraic varieties taking values in a ring of motives constructed from the category of algebraic varieties.
- Motivic integration as initially developed by Denef and Loeser (and then others) grounds Kontsevich's theory in the model theory of $\mathbb{Z}$-valued henselian fields allowing for rigorous transfers between different p-adic integration theories and even different characteristics.
- Motivic integration over algebraically closed valued fields in the sense of Hrushovski and Kazhdan is a theory in which the measures are defined on sets defined in algebraically closed valued fields and the values lie in (semi-)rings built from an amalgam of algebraic varieties and polytopes.

Why am I talking about motivic integration?

Given that my own connection to this subject is tangential and we have five more lectures scheduled on this subject to be delivered by the principal practitioners of motivic integration, why am I talking now?

Most of us here are aware that motivic integration is a deep subject and owes much of its efficacy to model theory, but even after nearly a decade and a half of conference presentations and seminars its aims and inner workings have not been internalized by model theorists. My goal in these lectures is to explain from the vantage of a model theorist who does not work actively on the subject

- what the main goals of motivic integration are,
- how the model theoretic integration theory hews closer to the geometric intuition, and
- how some of the technical constructions work.

Other introductions and surveys

The literature on motivic integration is vast and there are several good introductory articles. You may want to read some of the articles listed below while following the lecture series on motivic integration.

- Denef and Loeser, Motivic integration and the Grothendieck group of pseudo-finite fields, (ICM 2002). (on Denef's webpage)
- Gordon and Yaffe, An overview of arithmetic motivic integration. `arXiv:0811.2160`
- Hales, What is motivic measure? `arXiv:math.AG/0312229`
- Yin, Grothendieck homomorphisms in algebraically closed valued fields, `arXiv:0809.0473v1`
- Blickle's bibliography [through 2004]: `http://www.mabli.org/old/jet-bibliography.html`

Prehistory: equality of Betti numbers for birational Calabi-Yau manifolds

Theorem (Batyrev)

If X and Y are birational smooth projective algebraic varieties over $\mathbb{C}$ possessing nonvanishing volume forms, then $H^i(X,\mathbb{C}) \cong H^i(Y,\mathbb{C})$ for every $i \geq 0$.

- Choosing integral models, for $p \gg 0$, the volume forms give canonical p-adic measures.
- $\int_{X(R)} d\mu_\omega = \#X(\mathbb{F}_q)/q^n$ where R is an unramified extension of $\mathbb{Z}_p$, $\mathbb{F}_q$ is the residue field, $d\mu_\omega$ is the canonical measure and $n = \dim X$.
- Birationality implies that on a set of codimension at least two, the measures on X and Y agree.
- It follows that X and Y have the same ζ-functions at p, and, hence, the same Betti numbers.

Kontsevich's generalization

The hypothesis and conclusion of Batyrev's theorem concern complex algebraic varieties and the Euclidean topology, but the proof passes through p-adic integration and uses a comparison between ℓ-adic étale cohomology and Betti cohomology.

Kontsevich introduced to motivic integration in order to carry out a similar calculation geometrically without any special appeal to p-adic methods.

I will delay a description of Kontsevich's construction until after I have recalled another theorem predating the formal development of motivic integration, but one which brings model theory into the picture.

Prehistory: rationality of Poincaré series

Let $f(x_1, \ldots, x_m) \in \mathbb{Z}_p[x_1, \ldots, x_m]$ be a polynomial in many variables over the p-adic integers. Define

$$N_n := \#\{a \in (\mathbb{Z}_p/p^n\mathbb{Z}_p)^m : f(a) = 0\}$$

Theorem (Igusa)

The Poincaré series

$$P_{f,p}(T) := \sum_{n=0}^{\infty} N_n (p^{-m}T)^n \in \mathbb{Q}[[T]]$$

is a rational function of T.

Prehistory: Poincaré series as p-adic integral

Define

$$I_{f,p}(s) := \int_{\mathbb{Z}_p{}^m} |f(x)|_p^s dx$$

Proposition

$$P_{f,p}(p^{-m-s}) = \frac{1 - p^{-s} I_{f,p}(s)}{1 - p^{-s}}$$

Consequently, to prove rationality of $P_{f,p}(T)$, it suffices to prove rationality of $I_{f,p}(s)$ as a function of p^{-s}.

Prehistory: rationality of Igusa integrals via cell decomposition

- Igusa proved the rationality of $I_f(s)$ via resolution of singularities.
- Denef's proof uses p-adic cell decomposition.

Sketch:

- Using the coordinatization from the cell decomposition, we reduce to the case of (parametrized) one-variable polynomials.
- Using quantifier elimination, one integrates over simple sets on which $|f|_p$ behaves lines $|x|_p^r$ for some rational r.
- When $r = 0$, this is just a matter of computing the volume of a ball. When $r > 0$, this becomes a geometric series.

Prehistory: uniformity in p

Theorem

Let $f(x_1, \ldots, x_m) \in \mathbb{Z}[x_1, \ldots, x_m]$ be a polynomial over the ring of integers. There is a polynomial $Q(X, T) \in \mathbb{Z}[X, T]$ and a natural number d so that for all sufficiently large primes p, $P_{f,p}(T) = H_p(T)/Q(p, T)$ where H_p is a polynomial of degree d.

This uniformity result was proven using quantifier elimination by Macintyre and Pas independently.

$K_0(\mathscr{C})$

Let $\mathscr{C}$ be a small category. For X an object of $\mathscr{C}$, let $[X]$ be the isomorphism class of X.

The Grothendieck group of $\mathscr{C}$, $K_0(\mathscr{C})$ is the free abelian group on the isomorphism classes of objects of $\mathscr{C}$ modulo the subgroup generated by $[X] + [Y] - [X \coprod Y]$ as X and Y range through the objects of $\mathscr{C}$.

If $\mathscr{C}$ has coproducts, products, a terminal object and $\times$ distributes over $\coprod$ in the sense that $A \times (B \coprod C) \cong (A \times B) \coprod (A \times C)$ holds universally, then $K_0(\mathscr{C})$ is a ring with the multiplication determined by $[X] \cdot [Y] := [X \times Y]$.

Poor man's motives

Let k be a field and Var_k the category of algebraic varieties over k.

The ring of "poor man's motives" over k is the Grothendieck ring $K_0(\mathrm{Var}_k)$.

To be honest, we need to use *constructible* morphisms: a morphism $f : X \to Y$ is given by writing $X = \dot{\bigcup} X_i$ as a finite disjoint union of varieties (not necessarily closed in X) for which the restriction of f to X_i is a regular map of algebraic varieties.

Set $\mathbb{L} := [\mathbb{A}^1_k] \in K_0(\mathrm{Var}_k)$ and define a filtration on $K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$ by letting $F^j K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$ be the subgroup generated by elements of the form $[V]\mathbb{L}^{-m}$ with $\dim(V) + m \leq j$.

Kontsevich's value ring is $\widehat{\mathscr{M}_k}$, the completion of $K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$ with respect to this filtration.

Arc spaces

Let X be an algebraic variety over an algebraically closed field k.

For each natural number n, there is an algebraic variety $\mathscr{A}_n X$, the n^{th} arc space of X, for which $\mathscr{A}_n X(k) = X(k[t]/t^n)$, canonically. Taking projective limits, we may identify $X(k[[t]])$ with $\mathscr{A}_\infty X(k) := \varprojlim \mathscr{A}_n X(k)$.

There are natural projections $\pi_{n,m} : \mathscr{A}_n X \to \mathscr{A}_m X$.

For X smooth, we may cover X by open affines U for which $\mathscr{A}_{n+1} U \cong U \times \mathbb{A}^{dn}$ where $d = \dim X$.

Motivic measure

If X is an algebraic variety of dimension d over k, then there is a countably additive measure μ_X defined on the σ-algebra generated by the semialgebraic subsets of $X(k[[t]])$ and taking values in $\widehat{\mathscr{M}_k}$ determined by the condition that if $Y \subseteq \mathscr{A}_{n+1}X$ is a closed subvariety, then $\mu_X(\pi^{-1}_{\infty,n+1}Y) = [Y]\mathbb{L}^{-nd}$.

Motivic integrals

Via the usual measure theoretic definition of integration, we can now make sense of motivic integrals.

For f a regular function on X, the integral

$$\int_{X(k[[t]])} \mathbb{L}^{-\operatorname{ord}_t(f)} d\mu$$

plays the central role in Kontsevich's proof and can be regarded as the evaluation at $s = 1$ of a motivic Igusa integral.

Comparing Grothendieck rings

The ring $\widehat{\mathscr{M}_k}$ may be understood through its realizations.

Given a homomorphism $\psi : \widehat{\mathscr{M}_k} \to R$ to another ring, we may apply ψ to the result of an integral and thereby obtain an R-valued integral.

- If k is a finite field, then there is a natural map $K_0(\mathrm{Var}_k[\mathbb{L}^{-1}]) \to \mathbb{Q}$ given by $[X] \to \#X(k)$.
- Via an ultraproduct construction, the finite field counting functions induce nonstandard counting functions $K_0(\mathrm{Var}_{\mathbb{C}}) \to {}^*\mathbb{Z}$.
- By assigning to a smooth projective variety its Hodge polynomial: $[X] \mapsto \sum_{i,j} \dim H^i(X, \Omega^j_X) u^i v^j$, one obtains a homomorphism $\widehat{\mathscr{M}_{\mathbb{C}}} \to \mathbb{Z}[u, v]$.

Dwork's rationality theorem

For X an algebraic variety over a finite field $k = \mathbb{F}_q$, the zeta function of X is

$$Z(X, q, t) := \exp(\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n) \in \mathbb{Q}[[t]]$$

Theorem (Dwork)

$Z(X, q, t) \in \mathbb{Q}(t)$

Dwork's proof uses p-adic functional analysis.

Motivic ζ-function

If X is an algebraic variety over a field k, then

$$Z_{\mathrm{mot}}(X,t) := \sum_{n=0}^{\infty} [\mathrm{Sym}^n(X)] t^n \in K_0(\mathrm{Var}_k)[[t]]$$

When $k = \mathbb{F}_q$ is a finite field, there is a counting of points homomorphism $\chi_q : K_0(\mathrm{Var}_k) \to \mathbb{Z}$ which extends to the formal power series and $\chi_q(Z_{\mathrm{mot}}(X,t)) = Z(X,q,t)$

When X is defined over $\mathbb{Z}$, we know that for each power of a prime q that $Z(X,q,t)$ is a rational function. Is this explained motivically?

What does “rational function” mean?

Theorem (Poonen)

If $\mathrm{char}(k) = 0$, *then* $K_0(\mathrm{Var}_k)$ *is not a domain.*

Consequently, it is ambiguous as to what one means by “rational function” over $K_0(\mathrm{Var}_k)$. There are some reasonable definitions.

A formal power series $f \in R[[t]]$ over the commmutative ring R is rational if

- for every homomorphism $\psi : R \to K$ from R to a field, $\psi(f) \in K(t)$.
- there are polynomials $P(t)$ and $Q(t)$ in $R[t]$ so that f is the unique solution to $Q(t)X = P(t)$.
- the coefficients of f satisfy a nontrivial linear difference equation.

These definitions are not equivalent.

Counter-examples

Theorem (Larsen-Lunt)

With respect to any of the above definitions of rational, there are algebraic varieties over $\mathbb{C}$ whose motivic zeta functions are not rational.

This theorem does not rule out the possibility of a motivic explanation (and, hence, uniformization) of Dwork's theorem. The proof of irrationality does not work in rings in which $\mathbb{L}$ is invertible.

Where does model theory enter?

- At first, mainly through quantifier elimination/cell deomposition theorems used to evaluate integrals.
- Ax-Kochen-Ershov theorems yield uniformities across characteristics.
- Grothendieck (semi-)groups/rings of theories (and associated structures) give a good language for studying comparisons.
- With the Cluckers-Loeser relative theory and the Hrushovski-Kazhdan ACVF theory, definability replaces measure theory thereby making even the integration theory more "motivic."

$K_0(T)$

Given a (not necessarily complete) theory T in the language $\mathscr{L}$, $K_0(T)$ is the Grothendieck ring of the category of definable sets relative to T.

That is, we treat each formula $\varphi(x_1, \ldots, x_n) \in \mathscr{L}$ as an object and set

$$\begin{aligned} \mathrm{Mor}(\phi, \psi) \quad := \quad & \{[\theta(x; y)]_\sim : T \vdash (\forall x)(\forall y)[\theta(x; y) \to \phi(x) \& \psi(y)] \\ & \& (\forall x)[\phi(x) \to (\exists! y)\theta(x, y)\} \end{aligned}$$

where $\theta \sim \vartheta$ just in case $T \vdash \theta \leftrightarrow \vartheta$.

Remark

For $K_0(T)$ might not have a multiplicative identity if there is no definable singleton.

Comparison to geometric Grothendieck rings

- $K_0(\mathrm{ACF}_0) = K_0(\mathrm{Var}_{\mathbb{Q}})$
- More generally, if K is an algebraically closed field of characteristic zero and k is a subfield, then $K_0(\mathrm{Th}(K, +, \times, \{\underline{a}\}_{a \in k})) = K_0(\mathrm{Var}_k)$.
- There is a natural surjective map $K_0(\mathrm{Var}_{\mathbb{F}_p}) \to K_0(\mathrm{ACF}_p)$ with a nontrivial kernel (as observed by Ambrus Pal).
- $K_0(\mathrm{Th}(\mathbb{Z}, +, 0, 1, <)) = 0$ since $1 = [\{0\}] = [x \geq 0] - [x > 0] = [x \geq 0] - [x \geq 1] = 0$ as $x \mapsto x + 1$ is a definable bijection between $\{x : x \geq 0\}$ and $\{x : x \geq 1\}$.
- $K_0(\mathrm{Th}(\mathbb{Q}_p, +, \times, 0, 1)) = 0$ (Cluckers-Haskell)

K_0 as a functor

If $T \subseteq T'$, then there is a natural map $K_0(T) \to K_0(T')$.

If we expand the language, then there is a natural map $K_0^{\mathscr{L}}(T) \to K_0^{\mathscr{L}'}(T)$, where we have added a superscript to indicate the language under consideration. This construction is often used when $\mathscr{L}'$ is an expansion of $\mathscr{L}$ by constants and T' is T together with an atomic diagram in these new constants.

Languages for valued fields

We shall consider two basic languages for valued fields.

- (A variant of) the language of Denef-Pas, a three-sorted language, with sorts for the valued field, the residue field and the value group and for which we are given an angular component function $\mathrm{ac} : K^\times \to k^\times$ where K is the valued field, k is the residue field, and ac is a homomorphism extending the reduction map on the units.
- The language of additive-multiplicative congruences, a two-sorted language with a sort for the valued field and a second sort RV to be interpreted as $K^\times/(1+\mathfrak{m})$.

Denef-Pas language in detail

For a valued field (K, v) we interpret the Denef-Pas language as follows.

- The sort VF, the valued field, is interpreted as K in the language of rings.
- The sort k, the residue field, is interpreted as the residue field in the language of rings.
- The sort Γ, the value group, is interpreted as the value group in the language of ordered abelian groups,

There are functions $\pi : \mathrm{VF} \to \mathrm{k} \cup \{\infty\}$ and $v : \mathrm{VF} \to \Gamma$ for the residue and valuation maps.
There is also a function symbol $\mathrm{ac} : \mathrm{VF} \to \mathrm{k}$ for an angular component map. Note that the angular component usually requires an additional choice of structure.

More on the Denef-Pas language

To achieve quantifier elimination in the Denef-Pas language, even for the theories of such structures as $\mathbb{C}((t))$, it is necessary to first force quantifier elimination in the residue field and value group sorts.

Theorem

If (K, v, ac) is a henselian valued field with an angular component function of residue characteristic zero, then $\mathrm{Th}(K, v, \mathrm{ac})$ eliminates quantifiers relative to k and Γ in the Denef-Pas language.

When we understand the definable sets in the residue field and the value group, quantifier elimination takes a cleaner form. For example, when the value group is elementarily equivalent to $\mathbb{Z}$, "relative to Γ" means that we should include divisibility predicates, and when the residue field is algebraically closed there is nothing more to do.

Motivic measure in terms of definable sets

Kontsevich's motivic measures can be regarded as a functions which takes definable subsets of $\mathbb{C}[[t]]^n$ to (a completion of a localization of) $K_0(\mathrm{ACF}_{\mathbb{C}})$.

Remark

Motivic measures do not give rise to homomorphisms $K_0(\mathrm{Th}(C[[t]])) \to K_0(\mathrm{ACF}_{\mathbb{C}})$ as, for instance, the measures are not invariant under all definable bijections and are not even defined on unbounded definable sets.

Arithmetic motivic integration

Consider instead the case of X being an algebraic variety defined over a pseudofinite field k of characteristic zero.

Exactly as with the geometric motivic integration, one can define a measure μ_X on the definable subsets of $X(k[[t]])$ taking values in a completion of the Grothendieck ring of the theory of pseudofinite fields localized by $\mathbb{L} := [x = x]$.

The quantifier elimination theorem of Fried-Sacerdote allows one to describe the definable sets in pseudofinite fields in terms of Galois stratifications. The Ax-Kochen-Ershov theorems now give a precise sense in which the theories of p-adic integration converge to motivic integration.

Motivic integration as maps of Grothendieck semi-rings

One of the major innovations of the paper Cluckers & Loeser, *Constructible motivic functions and motivic integration*, `arXiv:math/0410203` is that motivic integrals are realized as homomorphisms from a semigroup of integrable constructible functions, axiomatically described, and via this construction the target ring is presented algebraically without a completion step.

This is a long and deep paper containing many other ideas and I think that the authors would rank their construction of a functorial relative theory of motivic integration as their principal achievement here.

Similar issues are addressed in Hrushovski & Kazhdan, *Integration in valued fields*, `arXiv:math/0510133` with similar though essentially different methods. Let me focus on the Cluckers-Loeser approach to integration as a homomorphism and focus on the role of the value group in the Hrushovski-Kazhdan theory.

The setting for Cluckers-Loeser logical motivic integration

- Fix a theory T of fields of characteristic zero possibly in a language expanded the language of rings. We write κ for the field generated by the constants.
- We consider the class of structures $k((t))$ in the three-sorted Denef-Pas language with divisibility predicates on $\mathbb{Z}$ as k ranges through the models of T.
- We consider the category GDef of pairs (ϕ, W) where W is itself a triple of the form $(\mathscr{X}, X, r)$ with $\mathscr{X}$ a variety over $\kappa((t))$, X a variety over k and $r \in \mathbb{N}$ and ϕ is a definable subset of $\mathscr{X}(\mathrm{VF}) \times X(\mathrm{k}) \times \mathbb{Z}^r$ (a "definable subassignment" in their language).

The ring $\mathbf{A}$

Let $\mathbf{A} := \mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}, \{\frac{1}{1-\mathbb{L}^n} : n \in \mathbb{Z}_+\}]$

If L is any field and $q \in L^\times$ is not a root of unity, then there is a ring homomorphism $\vartheta_q : \mathbf{A} \to L$ given by $\mathbb{L} \mapsto q$.

Define

$$\mathbf{A}_+ := \{a \in \mathbf{A} : \vartheta_q(a) \geq 0 \text{ for every real number } q > 1\}$$

Constructible motivic functions

For $S = (\varphi, (\mathscr{X}, X, r)) \in \mathrm{GDef}$, the ring $\mathscr{P}(S)$, of constructible Pressburger functions on S, is the subring of the ring of functions from φ to $\mathbf{A}$ generated by

- constant functions,
- the characteristic functions of definable subsets of φ,
- definable functions $\alpha : \varphi \to \mathbb{Z}$, and
- functions of the form $x \mapsto \mathbb{L}^{\beta(x)}$ where $\beta : \varphi \to \mathbb{Z}$ is definable.

$\mathscr{P}_+(S)$ is the sub-semi-ring of $\mathscr{P}(S)$ consisting of those functions all of whose values lie in $\mathbf{A}_+$.

Summability

We say that a sequence $\langle a_i \rangle_{i=0}^{\infty}$ of elements of $\mathbf{A}$ is summable if for each real number $q > 1$ the series $\sum_{n=0}^{\infty} |\vartheta_q(a_i)|$ converges.

Note that to say that the sequence is summable does not mean that $\sum_{n=0}^{\infty} a_i \in \mathbf{A}$.

Integration over $\mathbb{Z}^r$

For S a definable subassignment we say that $\varphi \in \mathscr{P}(S \times \mathbb{Z}^r)$ is S-integrable if for every point $s \in S$ the sequence $\langle \varphi(s;i) : i \in \mathbb{Z}^r \rangle$ is summable and write $I_S\mathscr{P}(S \times \mathbb{Z}^r)$ for the set of S-integrable functions.

Theorem

There is a unique homomorphism $\mu_S : I_S\mathscr{P}(S \times \mathbb{Z}^r) \to \mathscr{P}(S)$ so that for every $q > 1$, $s \in S$, and $\varphi \in I_S\mathscr{P}(S \times \mathbb{Z}^r)$

$$\vartheta_q(\mu(\varphi)(s)) = \sum_{i \in \mathbb{Z}^r} \vartheta_q(\varphi(s,i))$$

- Key observation: via quantifier elimination, reduce to computing (derivatives) of geometric series.
- Thus, when computing sums over definable sets in the value group, it is not necessary to perform a completion step.

Constructible motivic functions

For S a definable subassignment, RDef_S is the category of definable subassignments of $S \times k^n$ ($n \in \mathbb{N}$).

We define the semi-ring of positive constructible motivic functions on S to be

$$\mathscr{C}_+(S) := SK_0(\mathrm{RDef}_S) \otimes_{\mathbb{N}[\mathbb{L}-1]} \mathbf{A}_+$$

where by "SK_0" we mean the Grothendieck semi-ring.

Integration of constructible motivic functions

One can define the notion of a volume forms on S which have the shape

$$g\mathbb{L}^{-\operatorname{ord}_t(f)}|\omega|$$

where $g \in \mathscr{C}_+(S)$, $f : S \to \mathrm{VF}$ is definable, and ω is an algebraic volume form on the Zariski closure of S. As with Weil's p-adic measures, there are canonical volume forms on definable subassignments.

Integration against this canonical volume form gives a homomorphism of semi-rings $\mathscr{C}_+(S) \to SK_0(T) \otimes_{\mathbb{N}[\mathbb{L}-1]} \mathbf{A}_+$.

Motivic integration over algebraically closed valued fields

In the Hrushovski-Kazhdan theory, motivic measures are defined on definable sets in algebraically closed fields of residue characteristic zero.

The theory of nontrivially valued algebraically closed fields has quantifier elimination (say, in the three-sorted language of valued fields without the angular component). It follows that many of the results on motivic integration in ACVF may be specialized back to the Cluckers-Loeser theory.

However, in the case of ACVF motivic integration, the integrals take their values in Grothendieck (semi-)rings associated to RV-structures, not merely of the residue field.

Additive-multiplicative congruences in detail

For (K, v), the language of additive-multiplicative congruences has two sorts, VF to be interpreted as the valued field in the language of rings as before and RV to be interpreted as $K^\times/(1 + \mathfrak{m})$ where $\mathfrak{m} := \{x \in K : v(x) > 0\}$ the maximal ideal of the ring of integers in K and the sorts are connected via the natural quotient map $\mathrm{rv} : \mathrm{VF} \to \mathrm{RV}$.

There is more than one reasonable choice for the language on RV, but we take

- a binary function symbol for multiplication,
- a 2-place relation V to be interpreted as $V(x, y) \leftrightarrow v(\widetilde{x}) \leq v(\widetilde{y})$ for any $\widetilde{x}$ and $\widetilde{y}$ with $\mathrm{rv}(\widetilde{x}) = x$ and $\mathrm{rv}(\widetilde{y}) = y$, and
- a 3-place relation A defined by $A(x, y, z) \leftrightarrow (\exists \widetilde{x})(\exists \widetilde{y})[\mathrm{rv}(\widetilde{x}) = x \& \mathrm{rv}(\widetilde{y}) = y \& \mathrm{rv}(\widetilde{x} + \widetilde{y}) = z]$

More on additive-multiplicative congruences

- As with the Denef-Pas language, for Henselian fields of residue characteristic zero, one has elimination of quantifiers relative to RV.
- In mixed characteristic, the same holds provided that one includes higher RV-structures: $\mathrm{RV}_n(K) := K^\times/(1 + n\mathfrak{m})$ for all $n \in \mathbb{Z}_+$.
- The advantage of RV over the Denef-Pas language is that the passage from a one-sorted theory of a valued field to its RV-structure is always a definable expansion, but the Denef-Pas language requires a nontrivial choice of an angular component and thereby changes the class of definable sets.
- The problem is that $1 \longrightarrow k^\times \longrightarrow \mathrm{RV} \xrightarrow{v} \Gamma \longrightarrow 0$ is a definably nonsplit extension and the structure on RV might be somewhat unfamiliar.

Lifting from RV to VF

Of course, if $X \subseteq (\mathrm{VF}^\times)^m$ is a definable set, then $\mathrm{rv}(X) \subseteq \mathrm{RV}^m$ is definable in RV. In some sense, we can reverse this operation.

Let $\mathrm{VF}[n]$ be the category of definable sets $X \subseteq \mathrm{VF}^m \times \mathrm{RV}^\ell$ which admit finite-to-one definable maps to VF^n. The morphisms are definable maps.

Let $\mathrm{RV}[n]$ be the category of pairs (X, f) where $X \subseteq \mathrm{RV}^m$ is definable and $f : X \to \mathrm{RV}^n$ is definable and finite to one with morphisms $(X, f) \to (Y, g)$ being definable bijections.

For $(X, f) \in \mathrm{RV}[n]$ define

$$\mathbb{L}(X, f) := \{(x_1, \ldots, x_n, y) \in \mathrm{VF}^n \times X : (\mathrm{rv}(x_1), \ldots, \mathrm{rv}(x_n)) = f(y)\}$$

From RV to VF

Theorem

Writing $SK_0(\mathrm{RV}) := \bigoplus SK_0(\mathrm{RV}[n])$ *and* $SK_0(\mathrm{VF}) := \bigoplus SK_0(\mathrm{VF}[n])$, *the maps* $\mathbb{L} : \mathrm{RV}[n] \to \mathrm{VF}[n]$ *induce a surjective map of semi-rings* $\mathbb{L} : SK_0(\mathrm{RV}) \to SK_0(\mathrm{VF})$

Moreover, the kernel is generated by a single relation:

$$[1]_1 = [1]_0 + [(0,\infty)]_1$$

where

- $[1]_0 = [(\{1\}, 1 \mapsto *)] \in \mathrm{RV}[0]$
- $[1]_1 = [(\{1\}, 1 \mapsto 1)] \in \mathrm{RV}[1]$, and
- $[(0,\infty)]_1 = [(\{x \in \mathrm{RV} : v(x) > 0\}, \mathrm{id})] \in \mathrm{RV}[1]$

Similar results are true for variants of these categories including those with volume forms.

What is $SK_0(\mathrm{RV})$?

Theorem

Working over a model, $\mathrm{SK}_0(\mathrm{RV}) \cong \mathrm{SK}_0(\mathrm{k}) \otimes \mathrm{SK}_0(\Gamma)$.

- Dropping the hypothesis of a model as a base, $\mathrm{SK}_0(\mathrm{RV}) \cong \mathrm{SK}_0(\mathrm{RES}) \times_{\mathrm{SK}_0(\Gamma^{finite})} \mathrm{SK}_0(\Gamma)$ where RES is the subcategory of RV of sets internal to k and Γ^{finite} is the subcategory of finite definable sets.
- $\mathrm{SK}_0(\mathrm{k})$ may be understood in terms of algebraic varieties over the residue field while $\mathrm{SK}_0(\Gamma)$ encodes geometrical algebra, at least over a model.

What have I ignored?

Even for the works I have sketched, I have omitted most of the material ranging from fundamental technical issues (such as, how do we understand equality of the classes of definable sets over different parameter sets?) to major theorems (such as the construction of a relative theory of motivic integration). A couple highlights follow.

Motivic integration for more complicated functions

Exponential sums and the corresponding p-adic exponential integrals do not fit into the standard version of motivic integration. However, by expanding the target rings for the motivic integrals, it is possible to make sense of motivic exponential integrals.

Orbital integrals

One might study an algebraic group G acting (algebraically) on a variety X for which the action is geometrically transitive but over, say, $\mathbb{Q}_p$, breaks into several orbits. In representation theory, p-adic integrals over such orbits are important. From the geometric point of view, the orbits are not geometric objects, but they are definable sets.

In a spectacular paper of Cluckers, Hales, and Loeser (`arXiv:0712.0708v1`), it is shown that identities of orbital integrals proven over $\mathbb{F}_p((t))$ transfer to $\mathbb{Q}_p$ for $p \gg 0$. In particular, this holds for the so-called Fundamental Lemma of the Langland's Program.

Where else does logic have to say about motivic integration?

There are innumerable results about which I have been silent, in part due to time constraints and in part due to my ignorance of the subject. In the talks of the experts, we can expect far deeper theorems and explanations. Allow me to mention two directions which appear promising but little studied.

- p-adic integration has been used to prove rationality results about zeta functions for some finitely generated groups and elimination of imaginaries in valued fields have already been applied here.
- Other geometric objects, such as finite dimensional difference varieties, should be used to take the values of motivic integrals. This would give model theoretic sense in which a lifting of Frobenius is encoded in a motivic integral.