Groups in the theory of compact complex manifolds

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Manifolds as structures

Definition

If M is a complex manifold, then a subset $X \subseteq M$ is analytic if for any point $x \in M$ there are an open neighborhood $x \in U \subseteq M$ and a holomorphic function $f : U \to \mathbb{C}^m$ for which
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Definition

The structure $\mathscr{CM}$ is the multisorted structure having a sort $\underline{M}$ for each compact complex manifold M for which for each finite sequence of basic sorts, $\underline{M_1}, \ldots, \underline{M_n}$, and analytic subset $X \subseteq M_1 \times \cdots \times M_n$ there is a basic relation $\underline{X}$ on the product of sorts $\underline{M_1} \times \cdots \times \underline{M_n}$ to be interpreted by X.

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- $\mathscr{CCM}$ is $\aleph_1$-compact.
- The language we have set for $\mathscr{CCM}$ has cardinality $2^{\aleph_0}$. For some compact complex manifolds M it is possible to find a countable reduct $\mathscr{L}$ so that the $\mathscr{L}_M$-definable sets (*ie* with parameters from M) in all the Cartesian powers of M coincide with our original class of definable sets. In general, this is not possible.

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- Complex algebraic geometry lives in $\mathscr{CCM}$ in the sense that the complex projective line $\mathbb{P}^1(\mathbb{C})$ is a compact complex manifold and $\mathbb{C} = \mathbb{P}^1(\mathbb{C}) \setminus \{\infty\}$ is a definable set and the field operations are definable. Moreover, by Chow's Theorem the induced structure on $\mathbb{C}$ is just that of its field structure with all the elements named.

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- While $\mathscr{CCM}$ does not eliminate imaginaries, the natural expansion to $\mathscr{A}$, whose sorts are the compact complex analytic spaces, does.

The question

Question

Let $\mathscr{A}' \succeq \mathscr{A}$ be a model of the theory of $\mathscr{A}$. What groups are interpretable in $\mathscr{A}'$?

An answer

Theorem (Pillay-Scanlon)

If G is a group interpretable in $\mathscr{A}$, then there are a compact complex Lie group T, a linear algebraic group L over $\mathbb{C}$, and a definable maps $\iota : L \to G$ and $\pi : G \to T$ so that the sequence
$0 \longrightarrow L \xrightarrow{\iota} G \xrightarrow{\pi} T \longrightarrow 0$ *is exact.*

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Theorem (Aschenbrenner-Moosa-Scanlon;Scanlon)

If G is a group interpretable in $\mathscr{A}'$, then there are a definably compact group T, a linear algebraic group L over $\mathbb{C}'$, and definable maps $\iota : L \to G$ and $\pi : G \to T$ so that the sequence
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- MOOSA proved a nonstandard version of the Riemann Existence Theorem from which it follows that any field interpretable in $\mathscr{A}' \succeq \mathscr{A}$ is definably isomorphic to $\mathbb{C}' := (\mathbb{P}^1)^{\mathscr{A}'} \setminus \{\infty^{\mathscr{A}'}\}$.

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Definition

Let $\mathscr{A}' \succeq \mathscr{A}$. If M is a compact complex manifold, then by an analytic subset of $M^{\mathscr{A}'}$ we mean a set of the form $(f^{\mathscr{A}'})^{-1}\{b\}$ where $f : M \to B$ is a holomorphic map between compact complex manifolds and $b \in B^{\mathscr{A}'}$.

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Definition

By a meromorphic function $f : M \to N$ between the irreducible analytic sets M and N we mean an irreducible analytic subset $\Gamma_f \subseteq M \times N$ for which there is a Zariski open and dense subset $U \subseteq M$ with $\Gamma_f \cap (U \times N)$ being the graph of a definable function.

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Definition

By a definable manifold in $\mathscr{A}'$ we mean a set M given together with a finite covering $M = \cup_{i=1}^{n} V_i$ and bijections $\psi_i : V_i \to U_i \subseteq X_i$ between each V_i and Zariski open subsets $U_i \subseteq X_i$ of analytic sets for which the induced transition maps are meromorphic.

Complex analysis in nonstandard manifolds

Let $\mathscr{A}' \succeq \mathscr{A}$ be an elementary extension of $\mathscr{A}$.

Proposition

If G is any group interpretable in $\mathscr{A}'$, then G admits a unique (up to isomorphism) structure of a definable group manifold.

Interpretation in $\mathbb{R}_{an}$

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- The theory of $\mathbb{R}_{an}$ is o-minimal.
- Regarding $\mathbb{C}$ as $\mathbb{R}^2$ via the identification $z \mapsto (\mathrm{Re}(z), \mathrm{Im}(z))$, any complex analytic function may be seen as a pair of real analytic functions. Using compactness, we may interpret $\mathscr{CCM}$ and $\mathscr{A}$ in $\mathbb{R}_{an}$.

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So, when we say that T is definably compact, we mean that with its unique group manifold structure it is definably compact when regarded as being defined in a model of $\mathbb{R}_{an}$.

Classification of strongly minimal groups

Theorem (Pillay-Scanlon;Aschenbrenner-Moosa-Scanlon)

Let G be a strongly minimal group interpretable in $\mathscr{A}' \succeq \mathscr{A}$. Then either G is definably compact or G is definably isomorphic to the additive or multiplicative group of $\mathbb{C}'$.

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- Using a fairly long though elementary argument in point-set topology, one shows that there is a smooth, definably compact definable manifold $\overline{G}$ and an embedding $\iota : G \to \overline{G}$ which with respect to the group manifold structure on G expresses G as a Zariski open subset of $\overline{G}$.

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- If $\iota(G) = \overline{G}$, we are done. Otherwise, one expresses G as a linear algebraic group by considering its action on the infinitesimal neighborhood of a point on the boundary $\overline{G} \setminus \iota(G)$. $\square$

Composition series

Proposition

If G is a connected group of finitely Morley rank, then there is a composition series $\{1\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ of normal definable subgroups for which each quotient G_{i+1}/G_i is contained in the definable closure of a strongly minimal set together with a finite set.

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Let $X \subseteq G$ be strongly minimal. Possibly removing finitely many points, X is indecomposable. Translating, we may assume that X contains the origin.
By Zilber's Indecomposability theorem, N, the group generated by the conjugates of X is normal, definable, connected and generated in finitely many steps from finitely many of the conjugates, $X^{g_1}, \ldots, X^{g_m}$.

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By induction, there is a good composition series $\{1\} = \overline{G}_0 \triangleleft \overline{G}_1 \triangleleft \cdots \triangleleft \overline{G}_n = \overline{G}$. Let $G_0 := \{1\}$ and $G_{i+1} := \pi^{-1}\overline{G}_i$ for $i \geq 0$. $\square$

What is to be done?

- To prove the main theorem, it suffices to consider connected groups G interpretable in $\mathscr{A}'$.
- We are given a composition series $\{1\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ from the previous proposition.
- We need to show that we may choose the composition series so that for some N each of the quotients G_{i+1}/G_i is linear algebraic for $i < N$ and G_{i+1}/G_i is definably compact for $i \geq N$.
- We then argue that indeed G_N is linear algebraic and G/G_N is definably compact.

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Rearranging the sequence

Lemma

If $1 \longrightarrow K \longrightarrow H \longrightarrow A \longrightarrow 1$ is an exact sequence of definable groups in $\mathscr{A}'$ where K is definably compact and A is a one-dimensional linear algebraic group, then H is definably isomorphic to $H \times A$.

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The key to the proof of this lemma is a compactification theorem similar to the strongly minimal case. A subgroup of H isomorphic to A via the projection is then realized via the action of H on the boundary.

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From this lemma, one can drop the hypothesis that $\dim(A) = 1$. Using this observation repeatedly, we find the desired rearranged composition series.

Definable compactness of G/G_N

On general grounds, in any o-minimal structure an extension of a definably compact group by a definably compact group is definably compact. Hence, G/G_N is definably compact.

Linearity of G_N

Lemma

If $1 \longrightarrow N \longrightarrow H \longrightarrow K \longrightarrow 1$ *is an exact sequence of definable groups in* $\mathscr{A}'$ *where both* N *and* K *are linear algebraic, then* H *is linear algebraic.*

- Pillay's generalized socle theorem: If H is a group definable in $\mathscr{A}'$, $X \subseteq H$ is an irreducible subvariety and $S \leq H$ is the stabilizer of X, then the quotient X/S is bimeromorphic with an algebraic variety and
- another compactification argument.

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The proof of this lemma uses two ingredients:

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From this, it follows that G_N is linear, and, hence, the main theorem.

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Are there other theories of groups of finite Morley rank for which our compactification arguments make sense?