Model Theory and Differential Algebr

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Question: What is model theory?

Answer: Model theory is the study of *models*, structures which
formal languages.

Definition 1 *A signature σ is a quadruple $(\mathcal{C}, \mathcal{F}, \mathcal{R}, a)$ where* $\mathcal{C}$
$\mathcal{R}$ *are disjoint sets (called the* constant symbols, function symb
relation symbols, *respectively) and $a : \mathcal{F} \cup \mathcal{R} \to \mathbb{Z}_+$ is a funct*
assigns the arity *of a function symbol or relation symbol.*

Definition 2 *If $\sigma = (\mathcal{C}, \mathcal{F}, \mathcal{R}, a)$ is a signature, then a σ-struc*
nonempty set M given together with an interpretation of σ. Tha
each $c \in \mathcal{C}$ one is given some $c^M \in M$. For each $f \in \mathcal{F}$ one is
$f^M : M^{a(f)} \to M$. For each $R \in \mathcal{R}$ one is given $R^M \subseteq M^{a(R)}$

In most cases under consideration here, σ will be the signature
differential rings. That is, $\mathcal{C} = \{0, 1\}$, $\mathcal{F} = \{+, \cdot, \partial\}$, $\mathcal{R} = \varnothing$, an
$a(+) = a(\cdot) = 2$ while $a(\partial) = 1$. Our σ-structures will be diffe
rings and the symbols of σ will be interpreted in the usual way.

To each signature $\sigma = (\mathcal{C}, \mathcal{F}, \mathcal{R}, a)$ there is an associated (first-
formal language built from the symbols in $\mathcal{C} \cup \mathcal{F} \cup \mathcal{R}$, a set of
names $\{x_i : i \in \mathbb{N}\}$, symbols for logical Boolean operations $\wedge$,
and quantification over elements $(\exists x_i)$ and $(\forall x_i)$.

Definition 3 *If $\sigma = (\mathcal{C}, \mathcal{F}, \mathcal{R}, a)$ is a signature, then the set of defined by the following recursion.*

- *c is term for any constant symbol $c \in \mathcal{C}$.*
- *x_i is a term for any natural number $i \in \mathbb{N}$.*
- *$f(t_1, \ldots, t_n)$ is a term if $f \in \mathcal{F}$ is a function symbol with r and $t_1, \ldots, t_n$ are all terms.*

Definition 4 *If σ is a signature, then the set of formulas of lang*
associated to σ, $\mathcal{L}(\sigma)$, is defined by the following recursion.

- $t_1 = t_2$ *is a formula if t_1 and t_2 are σ-terms.*
- $R(t_1, \ldots, t_n)$ *is a formula if $R \in \mathcal{R}$ is a relation symbol wi* $n = a(R)$ *and $t_1, \ldots, t_n$ are all σ-terms.*
- $(\varphi \wedge \psi)$ *[read as "φ and ψ"] is a formula if φ and ψ are* *formulas.*
- $\neg(\varphi)$ *[read as "not φ"] is a formula if φ is a formula.*
- $(\exists x_i)(\varphi)$ *[read as "There exists x_i such that φ."] is a formu* *formula.*

If M is a σ-structure, then each formula in $\mathcal{L}(\sigma)$ has a natural i
in M.

If all the variables of the formula ψ are bound by a quantifier (s
formula is called a *sentence*), then M must decide the truth valu
We write $M \models \psi$ [read "M models ψ"] if M interprets ψ as tru

If T is a set of sentences, then we write $M \models T$ iff $M \models \psi$ for
$\psi \in T$.

Definition 5 *The* theory of M, $\mathrm{Th}(M)$, *is the set of all σ-senten
are true in M.*

A theory *is a set T of sentences for which there is some structur
$M \models T$.*

If some of the variables of ψ are free, then ψ defines a subset o
power of M. If the free variables of ψ are among $x_1, \ldots, x_n$, th

write $\psi(M) := \{(a_1, \ldots, a_n) \in M^n : M \models \psi(a_1, \ldots, a_n)\}$ wh
$\psi(a_1, \ldots, a_n)$ denotes the result of substituting a_i for the varia

Example 1 If σ is the signature of differential rings and R is a differential ring considered as a σ-structure in the natural way, $\varphi := (\exists x_2)(x_2 \cdot \partial(x_1) = 1)$, then $\varphi(R) = \{a \in R : \partial(a) \in R^\times\}$

Definition 6 *If M is a σ-structure and $A \subseteq M$, then we define be the language obtained by adjoining one new constant symbo $a \in A$ to σ. M has a natural $\mathcal{L}_A(\sigma)$-structure.*

Definition 7 *We say that the σ-structure N is an* elementary ex *M (written $M \preceq N$) if N is a model of the theory of M in $\mathcal{L}_M($*

In Weil's approach to the foundations of algebraic geometry, a c
is played by the notion of a universal domain: an algebraically c
field into which every "small" field of the same characteristic m
embedded and for which any isomorphism between "small" sub
may be extended to an automorphism.

Question 2 Is there an analogous notion of universal domain in
differential algebra?

For many natural theories there are no universal domains. How
Abraham Robinson arrived at a positive answer to Question 2 b
the *model completion* of the theory of differential fields of chara
zero.

Definition 8 *The theory T' is a model companion of the theory*

- *T and T' are co-theories: every model of T may be extend*
  model of T' and vice versa *and*
- *every extension of models of T' is elementary: if $M, N \models$*
  $M \subseteq N$, then $M \preceq N$.

If relative to T' every nonsentence is equivalent to a quantifier
formula, then T' is called a model completion *of T.*

If T has a model companion, then it has only one.

Example 3 • The theory of algebraically closed fields is the completion of the theory of fields.

- The theory of real closed fields is the model companion of of formally real fields. Considered with the signature $(\{0, 1\}, \{+, \cdot\}, \{<\})$ it is the model completion of the theor ordered fields.

Theorem 9 *The model completion of the theory of differential characteristic zero is the theory of differentially closed fields of characteristic zero,* DCF_0.

The fact that DCF_0 eliminates quantifiers takes a geometric for

Proposition 4 *If* $K \models \mathrm{DCF}_0$, $X \subseteq K^n$ *is Kolchin-constructible* $f : K^n \to K^m$ *is a differential rational function, then* $f(X) \subseteq$ *also Kolchin-constructible.*

There are a few reasonable ways to axiomatize DCF_0. Definitio
due to Lenore Blum.

Definition 10 *A differential field of characteristic zero* K *is dif
closed if for each pair* $f, g \in K\{x\}$ *of differential polynomials w
irreducible and* g *simpler than* f*, there is some* $a \in K$ *with* $f($
$g(a) \neq 0$.

Ehud Hrusovski provided geometric axioms. Before we can sta
geometric axioms, we need to recall the definition of jet spaces.

Definition 11 *If (K, ∂) is a differential field of characteristic z* *scheme over K, and $n \in \mathbb{N}$ is a natural number, then the n-th je* *X is the scheme $\nabla_n X$ which represents the functor $K - \partial -$* **Sc** *given on affines by $(R, \partial) \mapsto X_{R[\epsilon]/(\epsilon^{n+1})}(R[\epsilon]/(\epsilon^{n+1}))$ where* *into a scheme over $R[\epsilon]/(\epsilon^{n+1})$ via the map $x \mapsto \sum_{i=0}^{n} \frac{1}{n!}\partial^n(x$*

Concretely, if $X = \mathrm{Spec} K[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, then $\nabla_1 X = \mathrm{Spec} K[x_1, \ldots, x_n; x_1', \ldots, x_n']/(f_1, \ldots, f_m, df_x \cdot \vec{x}' -$ where g^∂ denotes the result of applying ∂ to the coefficients of

The reduction map $R[\epsilon]/(\epsilon^{n+1}) \to R[\epsilon]/(\epsilon^{m+1})$ corresponds t projection $\pi : \nabla_n X \to \nabla_m X$.

Proposition 5 *A differential field of characteristic zero K is di* *closed if and only if for any irreducible affine variety X over K* *Zariski constructible set $W \subseteq \nabla_1 X$ with $\pi \restriction_W : W \to X$ domin* *is some point $a \in X(K)$ with $(a, \partial a) \in W(K)$.*

The theory of differentially closed fields of characteristic zero i
transcendental theory.

Definition 12 *A theory T in the language $\mathcal{L}$ is* totally transcend
for every $M \models T$ every consistent $\mathcal{L}_M$ formula has ordinal valu
rank. *The Morley rank of a formula $\psi(\vec{x}) \in \mathcal{L}_M(\vec{x})$ is defined b*
following recursion.

- $\mathrm{RM}(\psi) = -1$ *if* $\psi(M) = \varnothing$
- $\mathrm{RM}(\psi) \geq 0$ *if* $\psi(M) \neq \varnothing$
- $\mathrm{RM}(\psi) \geq \alpha + 1$ *if there is some $N \succeq M$ and a sequence* {
  of $\mathcal{L}_N$-formulas such that $\varphi_i(N) \subseteq \psi(N)$ for each i,
  $\varphi_i(N) \cap \varphi_j(N) = \varnothing$ for $i \neq j$, and $\mathrm{RM}(\varphi_i) \geq \alpha$ *for all i*
- $\mathrm{RM}(\psi) \geq \lambda$ *for λ a limit ordinal if* $\mathrm{RM}(\psi) \geq \alpha$ *for all α* <
- $\mathrm{RM}(\psi) := \min\{\alpha : \mathrm{RM}(\psi) \geq \alpha$ *but* $\mathrm{RM}(\psi) \not\geq \alpha + 1\} \cup \{$

Totally transcendental theories carry many other ranks (Lascar,
local, *et cetera*). These ranks are all distinct in differentially clo

Many deep theorems have been proven about general totally
transcendental theories, but for all practical purposes, the theory
differentially closed fields is the only known mathematically sig
theory to which the deeper parts of the general theory apply.

Definition 13 *Let T be a theory, $M \models T$ a model of T and $A \subseteq$*
subset. A prime model *of T over A is a model $P \models T$ with $A \subseteq$*
having the property that if $\iota : A \hookrightarrow N$ is an embedding of A int
other model $N \models T$, then ι extends to an embedding of P into

Theorem 6 (Shelah) *If T is a totally transcendental theory, th*
model $M \models T$ and subset $A \subseteq M$ there is prime model over A.
the prime model is unique up to isomorphism over A.

Corollary 14 *If K is a differential field of characteristic zero, t*
is a differentially closed differential field extension K^{dif}/K, ca
differential closure of K, which embeds over K into any differe
closed extension of K and which is unique up to K-isomorphis

The theory of algebraically closed fields is also totally transcen
the prime model over a field K is its algebraic closure K^{alg}. Th
algebraic closure is also *minimal*. That is, if $K \subseteq L \subseteq K^{alg}$ wi
algebraically closed, then $L = K^{alg}$.

Theorem 7 (Kolchin, Rosenlicht, Shelah) *If K is a differenti*
of $\mathbb{Q}$, then there are $\aleph_0$ differentially closed subfields of K.

Trivial differential equations are responsible for Theorem . *Triv*
not mean *easy* or *unimportant*. Rather, it means that an associat
combinatorial geometry is degenerate.

Definition 15 *A combinatorial pregeometry is a set S given tog*
a closure operator $\mathrm{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$ *satisfying universally*

- $X \subseteq \mathrm{cl}(X)$
- $X \subseteq Y \Rightarrow \mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$
- $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$
- *if* $a \in \mathrm{cl}(X \cup \{b\}) \setminus \mathrm{cl}(X)$, *then* $b \in \mathrm{cl}(X \cup \{a\})$.
- *if* $a \in \mathrm{cl}(X)$, *then there is some finite* $X_0 \subseteq X$ *such that* a

If (S, cl) *satisfies* $\mathrm{cl}(\varnothing) = \varnothing$ *and* $\mathrm{cl}(\{x\}) = \{x\}$, *then we say tha*
a combinatorial geometry.

Example 8

- If S is any set and $\mathrm{cl}(X) := X$, then (S, cl) is a combinatorial geometry.
- If S is a vector space over a field K and $\mathrm{cl}(X) :=$ the K-sp
then (S, cl) is a combinatorial pregeometry.
- If S is an algebraically closed field and $\mathrm{cl}(A)$ is the algebra
of the field generated by A, then (S, cl) is a combinatorial pregeometry.

Definition 16 *The pregeometry* (S, cl) *is* trivial *if for any* $X \in$
has $\mathrm{cl}(X) = \bigcup_{x \in X} \mathrm{cl}(\{x\})$.

Definition 17 *If* (S, cl) *is a pregeometry, then a set* $X \subseteq S$ *is* in
if for any $x \in X$ *one has* $x \notin \mathrm{cl}(X \setminus \{x\})$.

Proposition 9 *If* (S, cl) *is a pregeometry,* $A \subseteq S$*, and* $X, Y \subseteq$
maximal independent subsets of A*, then* $\|X\| = \|Y\|$*. We define*
$\dim(A) := \|X\|$.

Definition 18 *A combinatorial pregeometry* (S, cl) *is* locally m
whenever $X, Y \subseteq S$ *and* $\dim(\mathrm{cl}(X) \cap \mathrm{cl}(Y)) > 0$ *we have*
$\dim(\mathrm{cl}(X) \cap \mathrm{cl}(Y)) + \dim(\mathrm{cl}(X \cup Y)) = \dim(\mathrm{cl}(X)) + \dim(\mathrm{cl}$

Definition 19 *Let M be a σ-structure for some signature σ. Le*
$\psi(x_1, \ldots, x_n)$ be some σ-formula with free variables among x_1
We say that the set $D := \psi(M)$ is strongly minimal *if $\psi(M)$ is*
and for any $N \succeq M$ and any formula $\varphi(x_1, \ldots, x_n) \in \mathcal{L}_N(\sigma)$ e
$\psi(N) \cap \varphi(N)$ is finite or $\psi(N) \cap (\neg\varphi)(N)$ is finite.

Definition 20 *Let M be a σ-structure for some signature σ. Le*
We say that $a \in M$ is model theoretically algebraic *over A if th*
formula $\psi(x) \in \mathcal{L}_A(\sigma)$ such that $M \models \psi(a)$ but $\psi(M)$ is finite
denote by $\operatorname{acl}(A)$ *the set of all elements of M which are algebra*

Example 10 If K is a differentially closed field and $A \subseteq K$, th
$\operatorname{acl}(A) = Q\langle A\rangle^{alg}$.

Proposition 11 *Let D be a strongly minimal set. Define*
$\mathrm{cl} : \mathcal{P}(D) \to \mathcal{P}(D)$ *by $X \mapsto$* $\operatorname{acl}(X) \cap D$. *Then* (D, cl) *is a con*
pregeometry.

Conjecture 21 (Zilber) *If D is a strongly minimal set whose a pregeometry is not locally modular, then D interprets an algebr closed field.*

Theorem 22 (Hrushovski) *Zilber's conjecture is false in gene*

Theorem 23 (Hrushovski, Zilber) *Zilber's conjecture holds f* geometries *(strongly minimal sets satisfying certain topologica smoothness properties.)*

Theorem 24 (Hrushovski, Sokolović) *Every strongly minima differentially closed field is a Zariski geometry after finitely ma are removed. Hence, Zilber's conjecture is true for strongly mi in differentially closed fields. In fact, if D is a non-locally modu strongly minimal set defined in some differentially closed field there is a differential rational function f for which $f(D) \cap K^{\partial}$ where $K^{\partial} := \{c \in K : \partial c = 0\}$.*

Theorem 24 is instrumental in the analysis of the structure of di
algebraic groups.

Theorem 25 (Hrushovski, Pillay) *Suppose that* $D_1, \ldots, D_n$ *a*
modular strongly minimal sets, G *is a definable group, and*
$G \subseteq \mathrm{acl}(D_1 \cup \cdots \cup D_n)$*. Then every definable subset of any po*
is a finite Boolean combination of cosets of definable subgroups

We call a group satisfying the conclusion of Theorem 25 *weakl*

Definition 26 *An* abelian variety *is a projective connected alge*
group. A semi-abelian variety is a connected algebraic group S
subalgebraic group T *which (over an algebraically closed field*
isomorphic to a product of multiplicative groups with S/T *bein*
abelian variety.

Theorem 27 (Manin, Buium) *If A is an abelian variety of dim defined over a differentially closed field of characteristic zero K there is a surjective differential rational homomorphism* $\mu : A(K) \to \mathbb{G}_a(K)^g$.

The kernel of μ is denoted by $A^\sharp$ and is called the *Manin kerne*

Theorem 28 (Buium, Hrushovski) *If A is an abelian variety over a differentially closed field K and A admits no non-zero a homomorphisms to abelian varieties defined over K^∂, then A(K weakly normal.*

Corollary 29 (Function field Manin-Mumford conjecture) *I*
abelian variety defined over a field K of characteristic zero, A
admit any nontrivial algebraic homomorphisms to abelian vari
defined over $\mathbb{Q}^{alg}$, and $X \subseteq A$ is an irreducible variety for whi
$X(K) \cap A(K)_{\mathrm{tor}}$ is Zariski dense, then X is a translate of an al
subgroup of A.

The function field Mordell-Lang conjecture follows from Theor
together with a general result of Hrushovski on the structure of
groups.

Definition 30 *If G is a group of finite Morley rank, then the* soc
G is the maximal connected definable subgroup of G for which
$G^\flat \subseteq \mathrm{acl}(D_1, \ldots, D_n)$ for some strongly minimal sets $D_1, \ldots,$

Example 12 If $G = A^\sharp$ is a Manin kernel, then $G^\flat = G$.

Definition 31 *Let G be a group defined over some set A. We sa*
rigid *if every subgroup of G is definable over* $\mathrm{acl}(A)$.

Example 13 If G is an abelian variety, then $G^\sharp$ is rigid.

Proposition 14 (Hrushovski) *Let G be a group of finite Morle*
Suppose that $G^\flat$ is rigid. If $X \subseteq G$ is a definable set of finite ra
trivial (generic) stabilizer, then X is contained (up to a set of lo
in a coset of $G^\flat$.

Theorem 15 (Buium, Hrushovski) *If G is a semiabelian vari*
over a differentially closed field K, $X \subseteq G$ is an irreducible su
$\Gamma \subseteq G(K)$ is a subgroup with $\dim_{\mathbb{Q}} \Gamma \otimes \mathbb{Q} < \infty$, and $X(K) \cap$
Zariski dense in X, then there is an algebraic subgroup $H \leq G$
algebraic group homomorphism $\psi : H \to H_0$ from H to an alg
group H_0 defined over the constants K^∂, an algebraic variety X
defined over K^∂ and a point $a \in G(K)$ such that $X = a + \psi^{-1}$

Remark Of course, a stronger form of Theorem 15 (due to Fal
Vojta, McQuillen, Bombieri, *et al*) in which one concludes that
translate of an algebraic subgroup of G holds.

As a consequence of the geometric axioms for differentially clo
Proposition 14, and intersection theory, Ehud Hrushovski and A
Pillay derived explicit bounds on the number of generic points o
subvarieties of semiabelian varieties.

Theorem 32 (Hrushovski, Pillay) *Let K be a finitely generate extension of $\mathbb{Q}^{alg}$. Let G be a semiabelian variety defined over Suppose that $X \subseteq G$ is an irreducible subvariety defined over $\mathbb{Q}$ cannot be expressed as $X_1 + X_2$ for some positive dimensional subvarieties X_1 and X_2 of G. If $\Gamma < G(K)$ is a finitely generat with $\Gamma \cap G(\mathbb{Q}^{alg})$ finite, then the number of points in $\Gamma \cap (X(K) \setminus X(\mathbb{Q}^{alg}))$ is finite and may be bounded by an expl function of geometric data.*

There is a general theory of *liaison* or *binding* groups in stable t
When specialized to the case of differentially closed fields, thes
groups give a differential Galois theory which properly extends
Picard-Vessiot and Kolchin stronlgy normal Galois theories.

Definition 33 *Let K be a differential field and X a Kolchin con set defined over K. Let $\mathcal{U} \supseteq K$ be a universal domain for differen closed fields extending K. A differential field extension $K \subseteq L$ called X-*strongly normal *if*

- *L is finitely generated over K as a differential field,*
- *$X(K) = X(L^{dif})$, and*
- *If $\sigma \in \mathrm{Aut}(\mathcal{U}/K)$ is a differential field automorphism of $\mathcal{U}$ then $\sigma(L) \subseteq L\langle X(\mathcal{U})\rangle$.*

The extension is called generalized strongly normal *if it is X-str normal for some X.*

Kolchin's strongly normal extensions are exactly the $\mathcal{U}^\partial$-stongl
extensions.

Theorem 16 (Pillay, Poizat) *If L/K is an X-strongly normal e then there is a differential algebraic group $G_{L/K}$ defined over K group isomorphism $\mu : \mathrm{Aut}(L\langle X(\mathcal{U})\rangle/K\langle X(\mathcal{U})\rangle) \to G_{L/K}(\mathcal{U})$ Moreover, there is a natural embedding $\mathrm{Aut}(L/K) \hookrightarrow \mathrm{Aut}(L\langle X(\mathcal{U})\rangle/K\langle X(\mathcal{U})\rangle)$ and with respect to th embedding we have $\mu(\mathrm{Aut}(L/K)) = G_{L/K}(K)$.*

As with Kolchin's differential Galois theory, we have a Galois correspondence between intermediate differential fields betwee and differential algebraic subgroups of $G_{L/K}$ defined over K.

Moreover, every differential algebraic group may be realized as
differential Galois group of some generalized strongly normal c
field extension. Thus, as every differential Galois group of a Ko
strongly normal extension is a group of constant points of an al
group over the constant and there are other differential algebrai
(Manin kernels, for example) differential Galois theory of gene
strongly normal extensions properly extends Kolchin's theory.

However, there are many finitely generated differential field ext
which are not generalized strongly normal. Trivial equations pr
phenomenon as well.

Definition 34 *Let X and Y be strongly minimal sets. Denote by $\pi : X \times Y \to X$ and $\nu : X \times Y \to Y$ the projections to X and respectively. We say that X and Y are non-orthogonal if there i infinite definable set $\Gamma \subseteq X \times Y$ such that $\pi \restriction_\Gamma$ and $\nu \restriction_\Gamma$ are fi functions.*

Theorem 24 may be restated as *If X is a non-locally modular st minimal set in a universal domain $\mathcal{U}$ for* DCF$_0$, *then $X \not\perp \mathcal{U}^\partial$.*

Theorem 25 together with a general group existence theorem of Hrushovski implies that if X is a nontrivial, locally modular, str minimal set in a differentially closed field, then X is non-orthog the Manin kernel of some simple abelian variety. Moreover, $A^\sharp$ and only if A and B are isogenous abelian varieties.

Question 17 How can one classify trivial strongly minimal set differentially closed fields up to nonorthogonality?

Question 18 Is there a structure theory for trivial strongly mini
in differentially closed fields analogous to the structure theory f
modular groups?

It is possible for a general trivial strongly minimal set to have n
whatsoever, but it is also possible for it to carry some structure.
example, the natural numbers $\mathbb{N}$ given together with the success
function $S : \mathbb{N} \to \mathbb{N}$ defined by $x \mapsto x + 1$ is a trivial strongly
set.

The answers to Questions 17 and 18 are unknown in general. In
particular, it is not known whether there is some trivial strongly
set X definable in a differentially closed field having a definabl
$f : X \to X$ with infinite orbits.

However, for *order one* trivial strongly minimal sets defined ov
constants, there are satisfactory answers to these questions.

Definition 35 *Let* $K \subseteq \mathcal{U}$ *be a countable differential subfield o universal domain. Let* $X \subseteq \mathcal{U}^n$ *be a constructible set defined ov define the* order *of* X *to be the maximum of* $\mathrm{tr.deg}_K K\langle x \rangle$ *as* x *r* X.

Definition 36 *Let* X *be a strongly minimal set defined over the say that* X *is* totally degenerate *if every permutation of* X *is ind element of* $\mathrm{Aut}(\mathcal{U}/A)$.

Theorem 19 (Hrushovski, Itai) *If* X *is a trivial order one set over the constants, then there is some totally degenerate* X' *wit*

Corollary 37 *Let* $f(x, y) \in \mathcal{U}^\partial[x, y]$ *be a nonzero polynomial constant coefficients. If* $\{a \in \mathcal{U} : f(a, a') = 0\} \perp \mathcal{U}^\partial$*, then the solutions to* $f(a, a') = 0$ *in a differential field* K *is bounded by of* $\mathrm{tr.deg}(K)$.

There has been significant development of the model theory of
fields of positive characteristic.

Carol Wood showed that the theory of differential fields of char
p admits a model companion DCF_p, the theory of differentially
fields of characteristic p.

However, differential fields satisfying fewer equations have pro
more useful. The theory of separably closed fields of finite imp
degree underlies Hrushovski's proof of the positive characterist
Mordell-Lang conjecture.

Differential algebra has played a crucial role in the model theor
analysis of well-behaved real-valued functions.

Definition 38 *An* o-minimal *expansion of* $\mathbb{R}$ *is a* σ*-structure on some signature* σ *having a binary relation symbol* $<$ *interprete usual manner such that for any* $\mathcal{L}_{\mathbb{R}}(\sigma)$*-formula* $\psi(x)$ *with one variable* x *the set* $\psi(\mathbb{R})$ *is a finite union of intervals and points.*

Example 20

- $\mathbb{R}$ considered just as an ordered set is o-mini [Cantor]
- $\mathbb{R}$ considered as an ordered field is o-minimal. [Tarski]

Theorem 21 (Wilkie) *The expansion of* $\mathbb{R}$ *by the field operatio exponential function is o-minimal.*

Behind the proof of Theorem 21 is another theorem of Alex Wi expansions of $\mathbb{R}$ by restricted Pfaffian functions.

Definition 39 *Let* $f_1, \ldots, f_n$ *be a sequence of differentiable re*
functions on $[0, 1]^m$*. We say that this sequence is a* Pfaffian cha
$\frac{\partial f_i}{\partial x_j} \in \mathbb{R}[x_1, \ldots, x_m, f_1, \ldots, f_i]$ *for each* $i \leq n$ *and* $j \leq m$*. We*
f *is a* Pfaffian function *if* f *belongs to some Pfaffian chain.*

Example 22 e^x restricted to the interval $[0, 1]$ is Pfaffian.

Theorem 23 (Wilkie) *If* $f_1, \ldots, f_n$ *is a Pfaffian chain, then* $(\mathbb{R}, +, \cdots, <, f_1, \ldots, f_n)$ *is o-minimal.*

Patrick Speisseger has generalized Wilkie's result to the case w
base structure is an arbitrary o-minimal exapansion of $\mathbb{R}$ rather
simply the real field.

Definition 40 *A* Hardy field *is a subdifferential field H of the g*
$+\infty$ *of smooth real-valued functions on the real line which is to*
ordered by the relation $f < g \Leftrightarrow (\exists R \in \mathbb{R})(\forall x > R)\, f(x) < g$

If $\mathcal{R}$ is an o-minimal expansion of $\mathbb{R}$, then the set of germs at $+$
$\mathcal{R}$-definable functions forms a Hardy field $\mathcal{H}(\mathcal{R})$.

Hardy fields carry a natural differential valuation with the valua
being the set of germs with a finite limit and the maximal ideal
set of germs which tend to zero.

Definition 41 *Let* (K, ∂) *be a differential field. A* differential v
on K *(in the sense of Rosenlicht) is a valuation* v *on* K *for whi*

- $v(x) = 0$ *for any nonzero constant* $x \in (K^{\partial})^{\times}$,
- *for any* y *with* $v(y) \geq 0$ *there is some* ϵ *with* $\partial(\epsilon) = 0$ *and* $v(y - \epsilon) > 0$, *and*
- $v(x), v(y) > 0 \Rightarrow v(\frac{y\partial(x)}{x}) > 0$.

Angus Macintyre, Dave Marker, and Lou van den Dries introdu
logarithmic-exponential series, $\mathbb{R}((t))^{LE}$, by closing $\mathbb{R}((t))$ und
logarithms, exponentials, and generalized summation.

$\mathbb{R}((t))^{LE}$ carries a natural derivation and differential valuation.

For all known examples $\mathcal{R}$ of o-minimal expansions of $\mathbb{R}$, there
natural embedding $\mathcal{H}(\mathcal{R}) \hookrightarrow \mathbb{R}((t))^{LE}$.

These embeddings, which may be regarded as divergent series
expansions, can be used to show that certain functions cannot b
approximated by other more basic function. In answer to a ques
Hardy, they show the following theorem.

Theorem 24 *The compositional inverse to* $(\log x)(\log \log x)$ *is
assymptotic to any function obtained by repeated composition o
semi-algebraic functions,* e^x*, and* $\log x$.

The empirical fact that many interesting Hardy fields embed int
$\mathbb{R}((t))^{LE}$ suggests the conjecture that the theory of $\mathbb{R}((t))^{LE}$ is
companion of the universal theory of Hardy fields.

Joris van der Hoeven has announced a sign change rule for diffe
polynomials over (his version of) $\mathbb{R}((t))^{LE}$. This result would g
way towards proving the model completeness of $\mathbb{R}((t))^{LE}$.

Matthias Aschenbrenner and Lou van den Dries have isolated a
ordered differential fields with differential valuations, H-fields,
every Hardy field belongs. They show, among other things, that
of H-fields is closed under Liouville extensions.

The model theory of valued differential fields serves as a frame
studying perturbed equations has also been developed.

Definition 42 *A* D-ring *is a commutative ring* R *together with*
$e \in R$ *and an additive function* $D : R \to R$ *satisfying* $D(1) =$
$D(x \cdot y) = x \cdot D(y) + y \cdot D(x) + eD(x)D(y)$.

If (R, D, e) is a D-ring, then the function $\sigma : R \to R$ defined b
$x \mapsto eD(x) + x$ is a ring endomorphism.

If $e = 0$, then a D-ring is just a differential ring. If $e \in R^\times$ is a
$Dx = \frac{\sigma(x)-x}{e}$ so that a D-ring is just a difference ring in disgui

Definition 43 *A valued* D*-field is a valued field* (K, v) *which i*
D*-ring* (K, D, e) *and satisfies* $v(e) \geq 0$ *and* $v(Dx) \geq v(x)$ *for*

Example 25 • If (k, D, e) is a D-field and $K = k((\epsilon))$ is th
Laurent series over k with D extended by $D(\epsilon) = 0$ and co
then K is a valued D-field.

- If (k, ∂) is a differential field of characteristic zero, $\sigma : k((\partial)) \to k((\epsilon))$ is the map $x \mapsto \sum_{i=0}^{\infty} \frac{1}{n!}\partial^n(x)\epsilon^n$, anc defined by $x \mapsto \frac{\sigma(x)-x}{\epsilon}$, then $(k((\epsilon)), D, \epsilon)$ is a valued D-
- If k is a field of characteristic $p > 0$ and $\bar{\sigma} : k \to k$ is any automorphism, then there is a unique lifting of $\bar{\sigma}$ to an autr $\sigma : W(k) \to W(k)$ of the field of quotients of the Witt vect Define $D(x) := \frac{\sigma(x)-x}{p}$, then $(W(k), D, p)$ is a valued D-f

Definition 44 *A valued D-field (K, v, D, e) is D-henselian if*

- *K has enough constants:* $(\forall x \in K)(\exists \epsilon \in K)\ v(x) = v(\epsilon)$ $D\epsilon = 0$ *and*
- *K satisfies D-hensel's lemma: if* $P(X_0, \ldots, X_n) \in \mathcal{O}_K[X_0$ *is polynomial with v-integral coefficients and for some* $a \in$ *integer i we have* $v(P(a, \ldots, D^n a)) > 0 = v(\frac{\partial P}{\partial X_i}(a, \ldots,$ *then there is some* $b \in \mathcal{O}_K$ *with* $P(b, \ldots, D^n b) = 0$ *and* $v(a - b) > 0$.

Theorem 26 *The theory of D-henselian fields with $v(e) > 0$, d ordered value group, and differentially closed residue field of characteristic zero is the model completion of the theory of equicharacteristic zero valued D-fields with $v(e) > 0$.*

There are refinements (with more complicated statements) of T with $v(e) \geq 0$ and restrictions on the valued group and residue

The relative theorem in the case of a lifting of a Frobenius on th
vectors may be the most important case.

Theorem 27 (Bélair, Macintyre, Scanlon) *In a natural expan
language of valued difference fields, the theory of the maximal*
extension of $\mathbb{Q}_p$ *together with an automorphism lifting the* p*-po*
Frobenius map eliminates quantifiers and is axiomatized by

- *the axioms for* D*-henselian fields of characteristic zero,*
- *the assertion that the residue field is algebraically closed o characteristic* p *and that the distinguished automorphism i* $x \mapsto x^p$*, and*
- *the assertion that the valued group satisfies the theory of* (Z *with* $v(p)$ *being the least positive element.*

Model theorists have also analyzed *difference algebra* in some

Definition 45 *A* difference ring *is a ring R given together with distinguished ring endomorphism $\sigma : R \to R$.*

Difference algebra admits universal domains in a weaker sense differential algebra.

Proposition 28 *The theory of difference fields admits a model companion,* ACFA. *A difference field $(K, +, \cdot, \sigma, 0, 1)$ satisfies and only if $K = K^{alg}$, $\sigma : K \to K$ is an automorphism, and fo irreducible variety X defined over K and irreducible Zariski co set $W \subseteq X \times \sigma(X)$ projecting dominantly onto X and onto $\sigma($ some $a \in X(K)$ with $(a, \sigma(a)) \in W(K)$.*

Unlike DCF_0, the theory ACFA is *not* totally transcendental, bu *supersimple*. In fact, the analysis of ACFA preceeded the devel the general theory of simple theories.

Zoé Chatzidakis, Ehud Hrushovski, and Ya'akov Peterzil have p
analogue of Theorem 24 for ACFA.

As a consequence of these theorems, Ehud Hrushovski derived
effective version of the Manin-Mumford conjecture.

While it is essentially impossible to actually construct different
closed fields, limits of Frobenius automorphisms provide mode
ACFA.

Theorem 29 (Hrushovski, Macintyre) *Let* $R := \prod_{n \in \omega, p \text{ prime}}$
$\sigma : R \to R$ *be defined by* $(a_{p^n}) \mapsto (a_{p^n}^{p^n})$*. If* $\mathfrak{m} \subseteq R$ *is a maxim*
for which $R/\mathfrak{m}$ *is not locally finite, then* $(R/\mathfrak{m}, \bar{\sigma}) \models$ ACFA.