Model theory, algebraic dynamics and local fields

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Descent theorem

Theorem (Chatzidakis, Hrushovski)

Let L be a finitely generated regular extension of the field K and (Y, f) a primitive AD over L. Then either (Y, f) constructibly descends to K or for each limited subset $S \subseteq X(L)$ there is a number $n := n(S)$ and a proper (not necessarily irreducible) subvariety $W \subsetneq Y$ so that if $a \in S \smallsetminus W(L)$, then at least one of $f(a), \ldots, f^n(a)$ is not an element of S.

- "Constructibly descends to K" means that there is an AD (Y_0, f_0) defined over K and a rational map of ADs $\gamma : (Y, f) \dashrightarrow (Y_0, f_0)_L$ which as a map of algebraic varieties is bijective on L^{alg} points. In characteristic zero, this is the same as being birational.
- The proof breaks into cases depending on whether or not (Y, f) is (generically) orthogonal to $(\mathbb{A}^1, \mathrm{id})$. If it is not, then it is essentially a translation variety: there is an algebraic group G acting transitively on Y and $g \in G(L)$ with $f(x) = g \cdot x$. This case is handled by Galois theory for difference equations. If $\deg(f) > 1$, then we are not in this case.

First step of proof: recollection of our conventions and hypotheses

- Recall that we have presented L as a countable directed union of K-points of constructible sets.
- Using this presentation, we may regard $Y(L)$ as $Y(L) = \bigcup \widetilde{Y}_d(K)$ where each $\widetilde{Y}_d$ is a constructible over K and this is a directed union.
- N.B.: We are not assuming that f respects this presentation of $Y(L)$. However, it will be the case that for each n there is some $m = m(n)$ so that $f(\widetilde{Y}_n(K)) \subseteq \widetilde{Y}_m(K)$ and the graph of the restriction of f to $\widetilde{Y}_n(K)$ is constructible.
- If the theorem fails and (Y, f) does not constructibly descend to K, then we are asserting that there is some $d \in \mathbb{Z}_+$ and a set $S \subseteq \widetilde{Y}_d(K)$ so that for any proper (not necessarily irreducible) subvariety $W \subsetneq Y$ and positive integer $n \in \mathbb{Z}_+$ there is a point $a \in S$ with $a, f(a), f^2(a), \ldots, f^n(a)$ all distinct elements of S.
- We will assume that we are in the case where $(Y, f) \perp (\mathbb{A}^1, \mathrm{id})$.

Second step: producing an AD over K which dominates (Y, f)

- If the theorem fails and (Y, f) does not constructibly descend to K, then we are asserting that there is some $d \in \mathbb{Z}_+$ and a set $S \subseteq \widetilde{Y}_d(K)$ so that for any proper (not necessarily irreducible) subvariety $W \subsetneq Y$ and positive integer $n \in \mathbb{Z}_+$ there is a point $a \in S$ with $a, f(a), f^2(a), \ldots, f^n(a)$ all distinct elements of S.

By the compactness theorem, we can find an elementary extension ${}^*K \succeq K$ so that letting ${}^*L$ be the field of fractions of $L \otimes_K {}^*K$, we have a non-pre-periodic point $a \in \widetilde{Y}_d(K)$ for which the forward orbit of a is contained in $W_d({}^*K)$ and is Zariski dense in Y.

Taking V to be the Zariski closure of the orbit a as a subset of $W_d({}^*K)$ and letting $g : V \to V$ be the restriction of f when read relative to the interpretation, $F : (V, g)_{{}^*L} \dashrightarrow (Y, f)_{{}^*L}$ is then dominant (where by F we mean the restriction of the interpretation map $F : \bigcup_{d=0}^{\infty} \widetilde{Y}_d({}^*K) \to Y({}^*L)$ to V)

From dominated to dominating

Using the fact that $K \preceq {}^*K$ is an elementary extension, we may assume that we actually have (V, g) defined over K and thus a dominant map $F : (V, g)_L \to (Y, f)$.

Working in a difference closed field (M, σ) extending (L^{alg}, id), we may regard F as the generic fibre of a rational map $\Phi : (V \times \mathbb{A}^1, g \times \mathrm{id}) \to (Y, f)$ over $(\mathbb{A}^1, \mathrm{id})$.

Constructibly quotienting by the equivalence relation on V defined by $v \sim v' \Longleftrightarrow \Phi(v, u) = \Phi(v', u)$ for generic u, we may assume that for $v \neq v'$, $\Phi(v, u) \neq \Phi(v', u)$ for generic u.

The function $\Phi(v, u)$ cannot really depend on the u variable as this would establish (generic) non-orthogonality between (Y, f) and $(\mathbb{A}^1, \mathrm{id})$.

Hence, on points, F is generically bijective.

What else might model theory say about dynamics?

- Variants of the trichotomy principle are valid for differential fields. While differential equations are not as closely linked to discrete algebraic dynamics, they do appear in analyses of dynamical systems preserving foliations, for instance. I do not know of an instance of a dynamically interesting set not coming from a group which is captured by well-behaved algebraic differential equations, but I would expect they exist.
- Differential and difference Galois theories for which the Galois groups are themselves definable groups may be developed from the model theoretic theory of liaison groups (or internal automorphism groups).
- There is a very well-developed model theory of valued fields including those equipped with distinguished automorphisms lifting the Frobenius from characteristic p to zero used to prove instances of the dynamical Manin-Mumford conjecture for periodic points for ADs lifting the Frobenius.
- I will concentrate on the model theory of “tame” real analysis, repeating some of what I said at the San Francisco AMS meeting.

Real analytic cyclic dynamical Mordell-Lang near an attracting fixed point

Theorem

Let $f_1, \ldots, f_\ell$ be a finite sequence of real analytic functions each defined on some interval. Let $a_1, \ldots, a_\ell$ be real numbers for which $\lim_{m\to\infty} f_i^{\circ m}(a_i)$ exists for each i. Then if $X \subseteq \mathbb{R}^\ell$ is a subanalytic set, the set $\{m \in \mathbb{Z}_+ : (f_1^{\circ m}(a_1), \ldots, f_\ell^{\circ m}(a_n)) \in X\}$ is a finite union of arithmetic progressions.

As we will see from the proof, the restriction to co-ordinatewise univariate dynamical systems is not necessary, though one does require a certain amount of analytic uniformizability.

Real analytic uniformizations

- If $f : \Delta \to \Delta$ is a complex analytic function from the disc $\Delta := \{x \in \mathbb{C} : |x| < 1\}$ back to itself for which $f(0) = 0$ but $\lambda := f'(0) \neq 0$ and $|\lambda| < 1$, then f is analytically conjugate the map $x \mapsto \lambda x$.
- More generally, if $f : \Delta^n \to \Delta^n$ is complex analytic, $f(0) = 0$, and the eigenvalues of df_0 are nonresonant, then f is analytically conjugate to its linearization.
- If $f : \Delta \to \Delta$ is a nonconstant complex analytic map and $f(0) = f'(0) = 0$, then f is conjugate to a power map $x \mapsto x^N$.
- Of course, for a general analytic function in several complex variables even with a fixed point there is no standard form.
- Working over $\mathbb{R}$, we should allow $-x^N$ as a standard form in the superattracting case.

It follows from the standard forms, that if $f(x)$ is real analytic and $f(0) = 0$, then possibly at the cost of replacing f with f^2, if a is close enough to 0, then there is a real analytic function E having the property that for large enough integers n we have $E(n) = f^n(a)$.

- For $f(x) = \lambda x$ with $0 < \lambda < 1$, we have $E(y) := \exp(y \log(\lambda))a$.
- For $f(x) = x^N$, we have $E(y) := \exp(\exp(y \log(N)) \log(a))$ as long as $a > 0$ which we may assume if N is even and we set $E(y) := -\exp(\exp(y \log(N)) \log(-a))$ for $a < 0$ if N is odd.
- In general, we must conjugate with the map expressing f in normal form.

Euclidean Skolem's method

Given the real analytic functions f_i and points a_i with $\lim f_i^n(a_i) = \overline{a_i}$, we compute uniformization function $E_i(y)$ as above (again, possibly after replacing f_i with f_i^2) so that for large integers $E_i(n) = f_i^n(a_i)$. Letting $f := (f_1, \ldots, f_\ell)$, we have $E(y) := (E_1(y), \ldots, E_\ell(y))$ uniformizing the orbit of $(a_1, \ldots, a_\ell)$ under f.

Given a real analytic function $G(x_1, \ldots, x_\ell)$, we would like to conclude that if there are infinitely many n with $G(E(n)) = 0$, then this holds for all n, but the compactness argument used in p-adic analysis fails here.

O-minimality

Definition

A first-order structure $(R, <, \cdots)$ is o-minimal if $<$ defines a total order on R and every definable subset of R (with parameters and all of the extra structure indicated by the ellipses) is a finite union of points and intervals.

- $(\mathbb{R}, <, +, \cdot, -, 0, 1)$ is o-minimal [Tarski]
- $(\mathbb{R}, <, +, \cdot, -, \exp, 0, 1)$ is o-minimal [Wilkie]
- $\mathbb{R}_{an}$, the real field augmented by function symbols for each f which is real analytic on $[-1, 1]^n$ (restricted to this box) for all n is o-minimal [van den Dries?]
- $\mathbb{R}_{an}(\exp)$ is o-minimal [van den Dries, Miller]

Our uniformization functions are explicitly definable in $\mathbb{R}_{an}(\exp)$. Hence, for any definable set X (in particular, an analytic set or a semianalytic or subanalytic set), if $\{y \in \mathbb{R} : E(y) \in X\}$ is infinite, it contains an interval of the form (b, ∞).

Extensions and limitations of this method

- The method applies whenever the orbit in question admits an $\mathbb{R}_{an}(\exp)$ uniformization.
- It gives nontrivial information (I will say more on the next slide) for semigroup actions.
- In general, it does not work well for complex dynamics as the uniformizing functions in this case involve oscillating functions.

Higher dimensional consequences of o-minimality

On the face of it, o-minimality asserts only simplicity of one-dimensional sets, but it has strong consequences on the structure of definable sets in any number of variables.

For example, it follows that every definable set in any number of variables may be decomposed into finitely many disjoint definable cells where each cell is definably homeomorphic to an open ball of some dimension. Moreover, the cell decomposition theorem holds uniformly.

The cell decomposition theorem yields strong uniformities on the topology of definable sets. For example, given a family $\{Y_b\}_{b \in B}$ of definable subsets of $\mathbb{R}^n$ for some o-minimal expansion of $(\mathbb{R}, <, +, \times, 0, 1)$, there is a uniform bound on the number of connected components of Y_b.

Number theoretically, the most relevant theorem is the Pila-Wilkie theorem on rational points.

Pila-Wilkie theorem

Theorem (Pila-Wilkie)

Let $X \subseteq \mathbb{R}^n$ be a set definable in some o-minimal structure on $\mathbb{R}$. We define X^{alg} to be the union of all the infinite connected semialgebraic subsets of X. Then for every $\epsilon > 0$ there is a constant C such that $\#\{a \in (X \setminus X^{\mathrm{alg}}) \cap \mathbb{Q}^n : H(a) \leq T\} \leq CT^\epsilon$

- This inequality (together with a non-commutative function field version of the Lindemann-Weierstrass theorem and Siegel's theorem) forms the basis of Pila's proof of the André-Oort conjecture.
- Applied to tuples of uniformization functions for dynamical systems, it gives numerical bounds for the dynamical Mordell-Lang problem for semigroups.

The end: What was the point?

- Compactness theorem
- Definability
- Orthogonality and the Zilber trichotomy
- O-minimality
- Model theory is a part of mathematics; use its results and ideas where you can; even better: develop parallel arguments using a formalism with which you are already comfortable but guided by the big picture and intuition generated by logic.