Nonstandard meromorphic groups

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Abstract

Extending the work of [7] on groups definable in compact complex manifolds and of [1] on strongly minimal groups definable in nonstandard compact complex manifolds, we classify all groups definable in nonstandard compact complex manifolds showing that if G is such a group then there are a linear algebraic group L, a definably compact group T, and definable exact sequence $1 \to L \to G \to T \to 1$.

Key words: compact complex manifolds, meromorphic groups, nonstandard analysis, restricted analytic functions

1 Introduction

Totally transcendental theories were isolated by Morley in the course of his proof of Łoś's conjecture on uncountably categorical theories [4]. Subsequent work instability theory produced a rich structure theory for the definable sets and types in models of totally transcendental theories. The natural examples of such theories coming from algebraic geometry and linear algebra do not exhibit the complicated structure envisioned by the general theory and while differential algebra supplies a fertile testing ground for algebraically minded model theorists most geometers find differentially closed fields exotic.

Totally transcendental theories are characterized by the Morley rank, an elaboration of the Cantor-Bendixson rank on the Stone spaces of types over models, being ordinal valued. In particular, a theory is totally transcendental if the Morley rank is always finite. Zilber observed that compact complex manifolds may be regarded as first-order structures of finite Morley rank [8]. Indeeed, the deep theory of Zariski geometries [2] and many of the nontrivial results of geometric stability theory apply in this context.

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A major theme in geometric stability theory is that to understand the structure of definable sets in a structure of finite Morley rank one should first understand the definable groups. Indeed, a structure of finite rank may be analyzed in terms of rank one sets and both the internal structure on the rank one sets and the way in which these sets form the analysis are described by definable groups. For now, let us simplify the definitions for the sake of exposition. (See [5] for a detailed treatement of analysis.) Morally, an analysis of a (type-)definable set X is given by a finite sequence of (type-)definable sets $X_0, \ldots, X_n = X$ and definable maps

$$X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_1} X_0$$

for which X_0 is finite and each fibre of $\pi_i : X_i \to X_{i-1}$ has rank one. The Zilber principle, which is true for definable sets in compact complex manifolds, asserts that for a rank one set either it is trivial, in the technical sense that all definable relations are reducible to binary relations, or there is a definable group entirely determining the structure of the given rank one set, and this group is either an algebraic group over an algebraically closed field or it is an abelian group without much additional structure. Moreover, the fibrations $\pi_i : X_i \to X_{i-1}$ are either essentially products or they are described by definable groups.

In [7] the groups interpretable in a compact complex manifold are completely described. While this result settles the issue of the possible nontrivial rank one definable sets in the standard model, it does not fully describe the groups which control the fibrations. Indeed, the groups implicated in the fibration $\pi_i : X_i \to X_{i-1}$ may themselves vary in a family. To understand such a family of groups is the same as to understand groups defined over a generic parameter. That is, one must describe groups in elementary extensions. The important case of strongly minimal, or, if you prefer, rank one, groups in nonstandard compact complex manifolds was handled in [1], but the problem of classifying general definable groups in nonstandard compact complex manifolds was left unresolved. In this paper, we dispose of this problem.

Depending on one's outlook, our results are either boring or reassuring. We show that, properly interpreted, the familiar structure theorem for groups definable in compact complex manifolds (itself a mild generalization of Chevallay's structure theorem for algebraic groups) holds for groups definable in nonstandard compact complex manifolds. Interestingly, as we shall explain later, this structure theorem would fail in the corresponding category of "compact complex manifolds" coming from nonstandard *real* analytic geometry and even in the category of complex Lie groups. Thus, while the theories of compact complex manifolds exhibit some of the pathologies of general theories of finite Morley rank, they are more regular than what is allowed even by the theory of higher dimensional Zariski geometries.

2 Compact complex manifolds as models

In this section we recall some of the basics of theories of finite Morley rank, the logical formalism for compact complex manifolds and discuss geometric interpretations of nonstandard models of their theories. The reader may wish to consult [3] for a fuller exposition of the model theory of compact complex manifolds.

In Morley's proof of the categoricity conjecture, Morley rank is introduced as a refinement of Cantor-Bendixson rank on the space of types. Working in a sufficiently saturated structure (as we shall) Morley rank may be defined succinctly by recursion. Let X be a definable (which for us always means with parameters) set, then

- $\mathrm{RM}(X) \geq 0$ if and only if $X \neq \varnothing$
- $\mathrm{RM}(X) \geq \alpha + 1$ if and on if there are infinitely many pairwise disjoint subsets $X_i \subsetneq X$ each with $\mathrm{RM}(X) \geq \alpha$
- $\mathrm{RM}(X) \geq \lambda$ (for λ a limit ordinal) if and only if $\mathrm{RM}(X) \geq \alpha$ for all $\alpha < \lambda$.

The Morley rank of X is α if $\mathrm{RM}(X) \geq \alpha$ but $\mathrm{RM}(X) \not\geq \alpha + 1$ and the Morley degree of X, $\mathrm{dM}(X)$, is d if there are d pairwise disjoint definable subsets $X_1, \ldots, X_d$ of X each with $\mathrm{RM}(X) = \mathrm{RM}(X_i)$ but one cannot find more than d such sets.

From the definition, one sees that X is finite and nonempty just in case $\mathrm{RM}(X) = 0$ and that in this case $\mathrm{dM}(X) = \#X$. A definable set of Morley rank and degree 1 is said to be *strongly minimal*. Equivalently, a definable set X is strongly minimal just in case it is infinite, but every definable subset is either finite or cofinite.

There are some standard examples of strongly minimal structures (*ie* models whose universes when considered as the set defined by $x = x$ are strongly minimal): a pure infinite set, an infinite set given together with a bijective unary function having no finite cycles, an infinite vector space over a field in the language of groups augmented by unary function symbols for scalar multiplication by each field element, and an algebraically closed field. The *Zilber principle* asserts that in a technical sense these examples exhaust the possibilities. The Zilber principle really asserts the existence of two important dividing lines in the class of strongly minimal sets. First, we say that a strongly minimal set X is *trivial* if the class of definable subsets of Cartesian powers of X is generated by the binary relations. Our first two examples above have this property while it is easy to see that no infinite group is trivial. Indeed, if a strongly minimal set is nontrivial, then this nontriviality must be witnessed by the presence of an interpretable infinite group. Secondly, we say that a group G considered as a first-order structure in some language possibly expanding the language of groups is *modular* if every definable subset of every Cartesian power of G is a finite Boolean combination of cosets of definable subgroups. The Zilber principle implies that if a strongly minimal group is not modular only if it interprets an algebraically closed field. Unfortunately, the Zilber principle (formerly, conjecture) does not hold for all strongly minimal sets, but it does hold under stronger hypotheses, especially in the case of *Zariski geometries*, strongly minimal structures satisfying certain topological conditions [2].

It follows on general grounds that any structure interpretable in a structure of finite Morley rank itself has finite Morley rank. In particular, if K is an algebraically closed field, the general linear group, $GL_n(K)$, of invertible $n \times n$ matrices over K has finite Morley rank as does any definable subgroup. The Cherlin-Zilber conjecture asserts that every simple group of finite Morley is a linear algebraic group. It is a fairly easy matter to deduce the Cherlin-Zilber conjecture from the Zilber principle. As the Zilber principle is true for the theory of compact complex manifolds (see the discussion below), no counter-example to the Cherlin-Zilber conjecture will be found amongst groups interpretable in compact complex manifolds. However, we shall use some of the techniques developed for the study of this conjecture. One of the most useful of these is the Zilber Indecomposability theorem. A definable subset $X \subseteq G$ of a group G is said to be *indecomposable* if for any definable subgroup $H \leq G$ of G either X/H is infinite or X/H is a singleton. The Zilber Indecomposability theorem says that if G is a group of finite Morley rank and $\mathcal{X}$ is a set of indecomposable subsets of G each of which contains the identity element, then the subgroup of G generated by $\mathcal{X}$ is expressible as $X_1 \cdot \cdots \cdot X_n$ for some finite sequence $X_1, \ldots, X_n$ from $\mathcal{X}$. Another more basic fact we use is that for every group G of finite Morley rank, there is a definable normal subgroup $G^\circ \trianglelefteq G$ for which G/G° is finite and G° is *connected*, that is, has no proper definable subgroups of finite index.

Recall that a complex manifold is a second countable Hausdorff space M admitting an open covering $M = \bigcup_{i \in I} U_i$ where for each index $i \in I$ we are given a homeomorphism $\vartheta_i : U_i \to V_i \hookrightarrow \mathbb{C}^{n_i}$ between U_i and some open set V_i in some power of the complex numbers for which the maps $\vartheta_i \circ \vartheta_j^{-1} : \vartheta_j(U_i \cap U_j) \to \vartheta_i(U_i \cap U_j)$ are holomorphic. We shall be interested (mostly) in compact complex manifolds, and for these we can choose the covering to have only finitely many charts. We say that a subset $X \subseteq M$ is *analytic* if it is closed and $\vartheta_i(U_i \cap X) \subseteq V_i$ is the common zero set of a sequence of holomorphic functions defined on V_i for each index $i \in I$. The analytic sets comprise the closed sets of the *analytic Zariski topology* on M.

Given a compact complex manifold M we construct a relational language $\mathcal{L}(M)$ and an interpretation of M as an $\mathcal{L}(M)$-structure as follows. For each natural number n and analytic set $X \subseteq M^n$ we have an n-ary relation symbol $\underline{X}(x_1, \ldots, x_n)$ which is to be interpreted as $\langle a_1, \ldots, a_n \rangle \in X \Leftrightarrow M \models \underline{X}(a_1, \ldots, a_n)$. Note that except in the case that M is finite, the language $\mathcal{L}(M)$ has cardinality $2^{\aleph_0}$ as every point in M is a closed subvariety, and thereby gives rise to a unary predicate. On occasion, one can find a countable sublanguage $\mathcal{L} \subseteq \mathcal{L}(M)$ so that every $\mathcal{L}(M)$-definable set (in any

Cartesian power of M) is parametrically definable in $\mathcal{L}$. For example, if $M = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ is the complex projective line, then it would suffice to take $\mathcal{L}$ to be generated by the closures in $(\mathbb{P}^1)^3$ of the graphs of addition and of multiplication. However, in general it is not possible to find such a countable sublanguage.

Working with multisorted first-order logic, we can consider all compact complex manifolds as a single structure, $\mathcal{A}$. For each compact complex manifold M (or, really, one for each isomorphism type), there is a sort, also denoted by M in $\mathcal{A}$. The basic relations on the production of sorts $M_1, \ldots, M_n$ and the relations $\underline{X}$ for the closed analytic subsets of $M_1 \times \cdots \times M_n$.

Zilber's theorem on quantifier elimination for compact complex manifolds lays the groundwork for the model theoretic investigation of $\mathcal{A}$.

Theorem 2.1 (Zilber) *The multisorted structure $\mathcal{A}$ eliminates quantifiers. Consequently, each compact complex manifold is $\aleph_1$-compact and has finite Morley rank with its Morley rank bounded by its complex dimension. Moreover, if X is a strongly minimal set definable in $\mathcal{A}$, then there is a cofinite subset $X' \subseteq X$ which is a Zariski geometry when one takes the traces of analytic sets on Cartesian powers of X' as the basic closed sets. Consequently, the Zilber principle holds in $\mathcal{A}$.*

The Zilber trichotomy in $\mathcal{A}$ takes a strong form as up to definable isomorphism the only interpretable field the field of complex numbers, given as the definable set $\mathbb{P}^1 \smallsetminus \{\infty\}$. Moosa has shown that in an elementary extension $\mathcal{A}' \succeq \mathcal{A}$ it remains the case that the only interpretable field (up to definable isomorphism) is the interpretation of $\mathbb{C}$ in $\mathcal{A}'$. By Chow's theorem, the analytic subsets of projective space are all already definable in the field language. There are some other complex spaces, *Moishezon spaces*, which are essentially algebraic. Recall that a *meromorphic function* $f : X \to Y$ from one (connected) complex manifold to another is not really a function at all but rather is given by its graph, $\Gamma_f \subseteq X \times Y$, an irreducible analytic subset of $X \times Y$ for which there is a Zariski dense and open subset $U \subset X$ so that the restriction of f to U is an analytic function. It follows from quantifier eliminaiton that every definable function in $\mathcal{A}$ is piece-wise meromorphic. A complex manifold M is *Moishezon* if there is a meromorphic function $f : M \to (\mathbb{P}^1)^n$ which is generically injective. We note that a product of Moishezon spaces is also Moishezon and that the image of a Moishezon space under a meromorphic function is also Moishezon. We shall use repeatedly the fact that a definable group which is Moishezon is definably isomorphic to an algebraic group.

We extend the geometric language to $\mathcal{A}' \succeq \mathcal{A}$, an elementary extension of $\mathcal{A}$. If $f : X \to Y$ is a holomorphic map between two compact compact manifolds and $f^{\mathcal{A}'} : X^{\mathcal{A}'} \to Y^{\mathcal{A}'}$ is the interpretation of this map in $\mathcal{A}'$, then each fibre of $f^{\mathcal{A}'}$ is, by definition, a closed analytic subset of $X^{\mathcal{A}'}$. It is a fact that these sets form the closed sets of a topology on $X^{\mathcal{A}'}$. In the standard model, for each integer d the set $X(f, d) := \{y \in Y \mid \dim(X_y) :=$

$\dim(f^{-1}\{y\}) = d\}$ is definable. in $\mathcal{A}'$ we define $\dim(f^{-1}\{y\}) = d \Leftrightarrow y \in X(f,d)^{\mathcal{A}'}$. Moreover, the set $S(f)$ of $y \in Y$ for which X_y is a manifold is also definable. In $\mathcal{A}'$ we say that X_y is a manifold just in case $y \in S(f)^{\mathcal{A}'}$.

In any first-order structure with a topology given by definable sets, one can make sense of the notion of a *definable manifold*: a Hausdorff space M given together with a finite open covering $M = \bigcup_{i=1}^n U_i$ and homeomorphisms $\vartheta_i : U_i \to V_i$ between each U_i and some definable open set V_i for which the induced transition maps are definable and continuous. Maps between definable manifolds are given by continous functions which when read in the charts are definable. We use implicitly the fact that any group interpretable in $\mathcal{A}$, or any elementary extension, has a unique structure of a definable manifold. It is important to note that a definable manifold in $\mathcal{A}$ need not be a compact complex manifold and it is not known whether or not any such definable manifold must be definably isomorphic to a submanifold of a compact complex manifold.

Since our language for $\mathcal{A}$ is too limited to discuss the usual Euclidean topology on compact complex manifolds, it is sometimes useful to work in an expanded language which can encompass this topology. The structure $\mathbb{R}_{an}$ is the ordered field of real numbers $(\mathbb{R}, +, \times, <, 0, 1)$ expanded by restricted real analytic functions. That is, for each natural number n and for each real power series $f = \sum a_\alpha X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ which is convergent in some neighborhood of the unit n-cube, we have an n-ary function symbol $\underline{f}$ which is interpreted as $\underline{f}(\boldsymbol{x}) = \sum a_\alpha \boldsymbol{x}^\alpha$ if $\boldsymbol{x}$ is in the unit cube and $\underline{f}(\boldsymbol{x}) = \overline{0}$ otherwise. The theory of $\mathbb{R}_{an}$ is o-minimal (meaning that every definable subset of the universe is a finite union of points and intervals) and has other nice properties. Using the usual interpretation of $\mathbb{C}$ as $\mathbb{R}^2$, one sees that every complex analytic function has real analytic real and imaginary parts and that every compact complex manifold, considered as a first-order structure as above, is interpretable in $\mathbb{R}_{an}$. Consequently, when studying elementary extensions of $\mathcal{A}$, possibly at the cost of taking a further elementary extension, we may assume that the structure is interpreted in a nonstandard model of $\mathbb{R}_{an}$.

We use the observation about the interpretability of $\mathcal{A}$ in $\mathbb{R}_{an}$ only for the purpose of giving a succinct definition of *definably compact*. If M is a definable manifold interpeted in some o-minimal expansion of a field, we say that M is *definably compact* if for every definable continuous functions $\gamma : [0,1) \to M$ the limit $\lim_{x \to 1} \gamma(x)$ exists. If M is an o-minimal expansion of $\mathbb{R}$, then *definably compact* and *compact* are equivalent, but, in general, definably compact need not imply compact. It is shown in [1] that a strongly minimal group interpretable in any model of the theory of $\mathcal{A}$ is either linear algebraic or definably compact. The reader may wish to consult that paper for a discussion of alternate characterizations of definable compactness.

3 Nonstandard Chevallay theorem

In this section we establish the structure theorem (Theorem 3.6 below) for groups definable in elementary extensions of $\mathcal{A}$. In what follows, we work in an elementary extension $\mathcal{A}' \succeq \mathcal{A}$ of $\mathcal{A}$. When we say "definable" we mean "definable in $\mathcal{A}'$, possibly with parameters."

The main step in proving our structure theorem for groups definable in $\mathcal{A}'$ is to show that a definable extension of a one dimensional linear algebraic group always (almost) splits. The corresponding result in the standard model does not transfer in any obvious way as the way in which one might witness the splitting may depend nonuniformly on parameters. Moreover, the proof presented in [7] of this result in the standard model does not work in out setting. We need three new ingredients for our proof. First, we use the stronger socle theorem of [6]. Secondly, we must form our partial compactifications definably. Finally, we use the nonstandard Fujiki embedding theorem from [1].

Lemma 3.1 *Suppose that G is a connected definable group, L is a linear algebraic group of dimension one, and that there is a normal definable subgroup $H \leq G$ for which G/H is definably isomorphic to L via the map $\pi : G \to L$. Then there is algebraic subgroup $A \leq G$ of dimension one for which $\pi(A) = L$.*

Proof. Working by induction on $\dim G$, we may assume that if $K < G$ is a proper definable subgroup, then $\pi(K)$ is trivial. Moreover, we may assume that G is not an algebraic group, for by the structure theory of algebraic groups over fields of characteristic zero, if G were an algebraic group, then such an A would exist.

Claim 3.2 *If $X \subsetneq G$ is a definable subset with $RM(X) < RM(G)$, then $\pi(X)$ is finite.*

Proof of claim: Working by induction on X, we may assume that X is closed and irreducible and that $\pi(X)$ is infinite (and, hence, cofinite in L). Under these hypotheses, $H \cdot X$ is generic in G, and thus finitely many left-translates of $H \cdot X$ cover G. Let $S \leq H$ be the stabilizer of X in H. If $S = H$, then as X and its stabilizer have the same dimension, X is a coset of H and we are done. So, we may assume that $S < H$.

By Corollary 2.8 of [6], X/S is Moishezon. Hence, by the above remark, $H \cdot (X/S)$ (and, therefore, also G/S) is Moishezon. Let $N := \bigcap_{g \in G} S^g$ be the intersection of all conjugates S in G. By Noetherianity, one may express N as $N = \bigcap_{i=1}^n S^{g_i}$ for some finite sequence $g_1, \ldots, g_n$ of elements of G. Moreover, by construction, it is clear that N is normal in G. The function $G/N \to \prod_{i=1}^n G/S^{g_i}$ given by the coordinatewise natural quotient maps is injective and definable. Hence, G/N, being identified with a definable subset of a Moishezon space is itself Moishezon, that is, an algebraic group. As $N \leq S < H$, the map π descends to a map $\nu : G/N \to L$. As G/N is algebraic and of dimension at least two, there is a proper subgroup $\overline{A} < G/N$ with $\nu(\overline{A}) = L$. Let $\widetilde{A}$ be the preimage of $\overline{A}$ in G under the quotient map $G \to G/N$. Then $\pi(\widetilde{A}) = L$

contradicting our reduction to the case that no proper subgroup of G maps onto L. ✠

Since G is definable, we can find a dense open set $U \subseteq G$ and a nonstandard connected manifold $\overline{U}$ for which $U \hookrightarrow \overline{U}$ identifies U with a Zariski open and dense subset of $\overline{U}$. We write $\overline{\pi} : \overline{U} \to \mathbb{P}^1$ for the meromorphic map on $\overline{U}$ extending π. Resolving the map and possibly replacing U with a subset, we may assume that $\overline{\pi}$ is holomorphic. The set $X := U \smallsetminus \pi^{-1}(\pi(U)) \subset G$ is a definable subset of G having strictly smaller Morley rank. Hence, by Claim 3.2 $\pi(X)$ is finite. Replacing U with $\pi^{-1}(\pi(U) \smallsetminus \pi(X))$, we may assume that for every $a \in \pi(U)$ that $\pi^{-1}\{a\} \subseteq U$.

We say that a component C of $\overline{U} \smallsetminus U$ is *vertical* if $\overline{\pi}(C)$ is a singleton and *horizontal* otherwise. Let U' be the complement in $\overline{U}$ of the horizontal components. Let $\pi' := \overline{\pi} \restriction U'$.

We construct a definable manifold G^* as follows. Let $S := \mathbb{P}^1 \smallsetminus L(= \{\infty\}$ ot $\{0, \infty\})$. As a definable set, $G^* := G \,\dot{\cup}\, \pi'^{-1}S$. We cover G^* with the open definable sets G and $\pi'^{-1}V_s$ where for $s \in S$ the set V_s is $\mathbb{P}^1 \smallsetminus ((L \smallsetminus \pi(U)) \cup (S \smallsetminus \{s\}))$. The transition maps given by the identifications in U are clearly definable and analytic. Let $\pi^* : G^* \to \mathbb{P}^1$ be the map defined by $\pi^*(x) = \pi(x)$ for $x \in G$ and $\pi^*(x) = \pi'(x)$ for $x \in \pi'^{-1}S$.

Multiplication $\mu : G \times G \to G$ extends to a meromorphic function $\mu^* : G^* \times G^* \to G^*$. Let B be the union of components of $G^* \smallsetminus G$ of codimension one in G^*. Using additivity of dimension, one sees that μ^* restricts to a meromorphic function $G^* \times B \to B$. As such, μ^* defines a generic action of G on B. In particular, if C is a codimension one component of $\pi^{*-1}\{\infty\}$ (such exists as $\dim \pi'^{-1}\{\infty\} = \dim H = \dim G - 1$), then generically, G acts definably on C.

As $\dim G > \dim C$, the generic stabilizer S in G of a generic point of C is infinite. By the nonstandard version of Fujiki's theorem (see Section 5 of [1]), the group S is linear algebraic. As L stabilizes $\{\infty\}$, we see that $\pi(S) = L$.□

In the case that the subgroup H is definably compact, we may drop the hypothesis that L is one dimensional.

Lemma 3.3 *Suppose that H is a definably compact group, L is a linear algebraic group, and G is connected definable group for which there is a definable exact sequence*

$$1 \longrightarrow H \longrightarrow G \xrightarrow{\pi} L \longrightarrow 1$$

Then there is a definable normal subgroup $K \trianglelefteq G$ with $\pi(K) = L$ and $K \cap H$ finite.

Proof. We work by induction on $\dim L$ with the case of $\dim L = 0$ being trivial. In the more general case, let $A \leq L$ be a one-dimensional algebraic subgroup. By Lemma 3.1 there is a definable one dimensional subgroup $\widetilde{A} \leq G$ with $\pi(\widetilde{A}) = A$. Let $\widetilde{N} \trianglelefteq G$ be the group generated by the conjugates of $\widetilde{A}$ in

G. As $\widetilde{A}$ is linear algebraic, so is $\widetilde{N}$. In particular, no infinite subgroup of $\widetilde{N}$ is definably compact. So, $\widetilde{N} \cap H$ is finite. We have an exact sequence

$$1 \longrightarrow H/(H \cap \widetilde{N}) \longrightarrow G/\widetilde{N} \stackrel{\nu}{\longrightarrow} L/\pi(\widetilde{N}) \longrightarrow 1$$

By induction, there is a subgroup $\overline{L} < G/\widetilde{N}$ with $\nu(\overline{L}) = L/\pi(\widetilde{N})$ and $\overline{L} \cap H/(H \cap \widetilde{N})$ finite. The preimage of $\overline{L}$ in G suits our purposes. □

As a consequence of Lemma 3.1 we see that even in nonstandard compact complex manifolds an extension of a linear algebraic group by a linear algebraic group is itself linear algebraic. It should be noted that even in the standard model an extension of an algebraic group by an algebraic group need not be algebraic.

Lemma 3.4 *Let H and L be linear algebraic groups and G a connected definable group for which there is a definable exact sequence*

$$1 \longrightarrow H \longrightarrow G \stackrel{\pi}{\longrightarrow} L \longrightarrow 1$$

Then G is also a linear algebraic group.

Proof. Working by induction on the dimension of G, we may assume that every proper subgroup of G is algebraic. We work now by induction on $\dim L$. Of course, if $\dim L = 0$, then the result is trivial. If $\dim L > 0$, then by the structure theory for linear algebraic groups there is a one-dimensional connected subgroup $A \leq L$. By Lemma 3.1 there is a definable one-dimensional subgroup $K < G$ with $\pi(K) = A$. By induction (or even just the fact that $\dim K = 1$) K is algebraic. Let N be the subgroup of G generated by the conjugates of K. By the Zilber Indecomposability Theorem N is generated in finitely many steps and is thus algebraic. If $N' := \pi(N)$, then we have an exact sequence $1 \to N \to G \to L/N' \to 1$. As $A \leq N'$, $\dim L/N' < \dim L$. Thus, G is algebraic. □

From the structure theorem for strongly minimal groups definable in $\mathcal{A}'$ we conclude that every definable group admits a composition series where each factor is either definably compact or linear algebraic.

Lemma 3.5 *Let G be a connected group definable in $\mathcal{A}'$. There is a finite sequence of normal subgroups $\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ such that for each i the quotient G_{i+1}/G_i is either linear algebraic or definably compact.*

Proof. We prove this by induction on $\dim G$ with the case of $\dim G = 0$ being trivial. More generally, let $A \leq G$ be a normal definable subgroup of G generated the conjugates of a strongly minimal subset. By the main theorem of [1], either A is an algebraic group or A is definably compact. If A is algebraic and L, the maximal linear algebraic subgroup of A, which by the usual Chevallay theorem is characteristic in A, is nontrivial, set $G_1 := L$. Otherwise, take $G_1 := A$. By induction, the quotient G/G_1 admits a good composition series whose preimage in G gives the desired composition series for G. □

With these lemmata in place we may finish the proof of our main theorem on the classification of groups definable in $\mathcal{A}'$.

Theorem 3.6 *Let G be a connected group definable in $\mathcal{A}'$. Then there are a connected linear algebraic group L and a definably compact group T fitting into a definable exact sequence*

$$1 \longrightarrow L \longrightarrow G \longrightarrow T \longrightarrow 1$$

Moreover, the group L is uniquely determined.

Proof. For the "moreover" clause, we note that if L and M are two connected linear algebraic subgroups of G, then the group generated by L and M in G is connected and algebraic and therefore linear by the corresponding fact for algebraic groups. Thus, the L of the statement of this theorem is *the* maximal connected linear algebraic subgroup of G.

As usual, we prove the theorem by induction on $\dim G$ with the case of $\dim G = 0$ being trivial. Let $G_1 \trianglelefteq G$ be the subgroup given by Lemma 3.5 and set $H := G/G_1$. By induction, if $M \trianglelefteq H$ is the maximal connected linear algebraic subgroup of H, then M is normal and $H/M =: S$ is definably compact. Let $\widetilde{M} \leq G$ be the connected component of the preimage of M in G. We have an exact sequence

$$1 \longrightarrow G_1 \longrightarrow \widetilde{M} \stackrel{\nu}{\longrightarrow} M \longrightarrow 1$$

If G_1 is linear algebraic, then by Lemma 3.4 the group $\widetilde{M}$ is also linear algebraic and the quotient $G/\widetilde{M} = S$ is definably compact as desired. On the other hand, if G_1 is not linear algebraic, then it is definably compact and Lemma 3.3 produces a normal linear algebraic subgroup $K \trianglelefteq \widetilde{M}$ with $\nu(K) = M$ and $K \cap G_1$ finite. Set $L := K$. The quotient G/L is an extension of a definably compact group $(G_1/(L \cap G_1))$ by another definably compact group (S) and is therefore a definably compact group itself.

Thus, in either case, G satisfies the conclusion of the theorem. □

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