p-Jets and Uniform Unramified Manin-Mumford

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The Uniform Unramified Manin-Mumford Theorem

Theorem 1 *Let R be a complete, mixed characteristic, discrete valuation ring with an algebraically closed residue field of characteristic p. Let $A \to B$ be a family of abelian schemes over R and $X \subseteq A$ a closed subscheme.*

There is a natural number n and a sequence of families of subabelian schemes $A^{(1)}, \ldots, A^{(n)} \subseteq A$ for which given any point $b \in B(R)$ there are an index set $I \subseteq \{1, \ldots, n\}$ and points $\zeta_1, \ldots, \zeta_n \in A_b(R)$ such that

$$X_b(R) \cap A_b(R)_{\text{tor}} = \bigcup_{i \in I} \zeta_i + A_b^{(i)}(R)_{\text{tor}}$$

Modularity and Groups of Lang-type

Definition: Let G be a commutative algebraic group over some algebraically closed field K. We say that $\Gamma \leq G(K)$ is of *Lang-type* if for any natural number n and subvariety $X \subseteq G^n$ of G^n the set $X(K) \cap \Gamma^n$ is a finite union of cosets of subgroups.

Proposition 2 (Pillay) *A group Γ is of Lang-type if and only if the structure $\mathcal{K} = (K, +, \cdot, \Gamma, \{\underline{a}\}_{a \in K})$ is stable and in $\mathcal{K}$ the set defined by Γ is one-based.*

Finiteness and Uniformity

Corollary 3 *If the subgroup $\Gamma \leq G(K)$ of the algebraic group G is of Lang-type, then for any family $\{X_b\}_{b \in B}$ of subvarieties of G there are a natural number n and algebraic subgroups $G_1, \ldots, G_n \leq G$ such that for any $b \in B$ there are a set $I \subseteq \{1, \ldots, n\}$ and points $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that*

$$X_b(K) \cap \Gamma = \bigcup_{i \in I} \gamma_i + (G_i(K) \cap \Gamma)$$

Witt vectors

- W is a functor from the category of characteristic p fields to complete discrete valuation rings with maximal ideal generated by p.
- As a set, $W[k] = W_{p^\infty}[k]$ is ${}^{\omega}k$.
- The residue field $W[k]/pW[k]$ is canonically isomorphic to k. More generally, the association $k \Rightarrow W[k]/p^n W[k]$ defines a ring scheme with $W[k]/p^n W[k]$ identified (as a set) with ${}^{n}k$.
- $W[\mathbb{F}_p] = \mathbb{Z}_p$, $W[\mathbb{F}_p^{\text{alg}}] = \mathbb{Z}_p^{\text{unr}}$, the maximal unramified extension (or the completion of the maximal unramified algebraic extension) of the p-adic integers.
- $\text{Aut}(k) = \text{Aut}_{\text{cont}}(W[k])$. In particular, the Frobenius automorphism lifts to the relative Frobenius on the Witt vectors.

p-Derivations

Definition: A *p-derivation* on a unital commutative ring R is a function $\delta : R \to R$ satisfying

- $\delta(x + y) = \delta(x) + \delta(y) + \Phi_p(x, y)$ where $\Phi_p(X, Y) \in \mathbb{Z}[X, Y]$ is the polynomial $\frac{1}{p}((X + Y)^p - X^p - Y^p)$ and
- $\delta(x \cdot y) = x^p \delta(y) + y^p \delta(x) + p\delta(x)\delta(y)$.

If $\tau : W[k] \to W[k]$ is the relative Frobenius, then $\delta_p : W[k] \to W[k]$ defined by $\delta(x) = \frac{\tau(x) - x^p}{p}$ is a p-derivation.

p-Jets

Definition: An *analytic function* $f : W[k]^n \to W[k]$ is a function of the form $x \mapsto \sum a_\alpha x^\alpha$ where $v(a_\alpha) \to \infty$ as $|\alpha| \to \infty$.

Definition: A *p-jet function* $g : W[k]^m \to W[k]$ is a function of the form $x \mapsto f(x, \ldots, \delta^\ell x)$ for some natural number ℓ and analytic function f of $m(\ell + 1)$ arguments.

Buium-Manin homomorphisms

Theorem 4 (Buium) *Let A be an abelian scheme over $W[k]$, the Witt vectors of an algebraically closed field of positive characteristic. There is a surjective p-jet group homomorphism $\mu : A(W[k]) \to \widehat{\mathbb{G}_a}^g(pW[k])$.*

- $A(W[k])_{\mathrm{tor}} \leq A^\sharp(W[k])$.
- $A^\sharp(W[k])/p^\infty A(W[k])$ is a finitely generated $\mathbb{Z}_p$-module.
- The reduction map $\pi : A(W[k]) \to \overline{A}(k)$ is surjective when restricted to $A^\sharp$.

We will prove the uniform unramified Manin-Mumford theorem by analyzing the following extension as a model-theoretic cover:

$$0 \longrightarrow A_0^\sharp \longrightarrow A^\sharp \longrightarrow \overline{A} \longrightarrow 0$$

Witt Vectors as Analytic Difference Rings

We consider $W[k]$ (for k an algebraically closed field of characteristic p) in a language having

- an n-ary function symbol f for each analytic function of n-variables $f : W[k]^n \to W[k]$,
- a binary function symbol $\mathcal{Q}$ interpreted as $\mathcal{Q}(x, y) = \frac{x}{y}$ if $vx \geq vy \neq \infty$ and $\mathcal{Q}(x, y) = 0$ otherwise,
- a unary function symbol σ interpreted as the relative Frobenius,
- a unary function symbol ac_n (one for each positive integer n) taking values in the imaginary sort $W[k]/p^n W[k]$ whose interpretation is defined by the relation $x(1 + p^n W[k]) = p^{vx}\text{ac}_n(x)$, and
- unary predicates D_n interpreted as $D_n(x) \Leftrightarrow n|vx$.

Basic Theorems about the Witt Vectors as Analytic Difference Rings

Theorem 5
- *$W[k]$ eliminates quantifiers as an analytic difference ring.*
- *If $k \subseteq k'$ is an extension of algebraically closed fields of characteristic p, then $W[k] \preceq W[k']$ as analytic difference rings.*
- *If $R \succeq W[k]$ is a saturated elementary extension and $\overline{\rho} : R/pR \to R/pR$ is an automorphism, then $\overline{\rho}$ lifts to an automorphism $\rho : R \to R$.*

Orthogonality in Analytic Difference Fields

Definition: We say that the definable sets X and Y are *orthogonal*, $X \perp Y$, if every definable subset of $X \times Y$ is a finite Boolean combination of sets of the form $A \times B$ where $A \subseteq X$ and $B \subseteq Y$ are definable sets.

In $W[k]$ considered as an analytic difference ring, the residue field and value group are orthogonal.

p-Jet Version of Uniform Unramified Manin-Mumford

Theorem 6 *Let $R \succeq W[k]$ be an elementary extension of the Witt vectors of an algebraically closed field k of positive characteristic considered as an analytic difference ring. If A is an abelian scheme over R and $X \subseteq A$ is a closed subscheme, then $X(R) \cap A^\sharp(R)$ is a finite union of sets of the form $Y(R) + B(R)$ where $Y \subseteq A_0^\sharp$ is a definable subset of $A_0^\sharp$ and and $B \leq A^\sharp$ is a definable subgroup of $A^\sharp$.*

Theorem 1 follows via a (non-trivial) compactness argument.

Definable Sets in Buium-Manin Kernels

Proposition 7 *Let $k \models \mathrm{ACF}_p$ be an algebraically closed field of characteristic p and let $R \succeq W[k]$ be an elementary extension of the analytic difference ring $W[k]$. If A is an abelian scheme over R and $X \subseteq A^\sharp(R)$ is a definable set, then X is a finite Boolean combination of sets of the form $Y(R) + \pi \restriction_B^{-1} Z(k)$ where $Y \subseteq A_0^\sharp$ is a definable subset of the kernel of reduction, $B \leq A^\sharp$ is a a definable subgroup, and $Z \subseteq \overline{A}$ is a subvariety of the special fibre.*

Key Ingredients of the Proof

- Reduce to the case that $R = W[\mathbb{F}_p^{\mathrm{alg}}]$.
- Mimic the proof of the socle theorem using the proalgebraic structure of $A^\sharp(W[k])$ to replace arguments that should only work in the finite Morley rank context. [This gives Proposition 7.]
- In passing from Proposition 7 to Theorem 6, one reduces to the case that any set of the form $\pi \restriction_B^{-1} Z(k)$ is actually the intersection of a subscheme of A with a subgroup B of $A^\sharp$ which maps finite-to-one to $\overline{A}$. In the case that $k = \mathbb{F}_p^{\mathrm{alg}}$, $B(R)$ is torsion so that by Raynaud's Theorem, the set under consideration is a finite union of cosets of groups.

Open Questions

- Does the uniformity theorem hold for the full torsion group over $\mathbb{C}$?
- What is the possible structure on $A_0^\sharp$?
- Can the appeal to Raynaud's Theorem be replaced with a Zariski geometry argument?