LIKELY INTERSECTIONS

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Abstract. We prove a general likely intersections theorem, a counterpart to the Zilber-Pink conjectures, under the assumption that the Ax-Schanuel property and some mild additional conditions are known to hold for a given category of complex quotient spaces definable in some fixed o-minimal expansion of the ordered field of real numbers.

For an instance of our general result, consider the case of subvarieties of Shimura varieties. Let S be a Shimura variety. Let $\pi : D \to \Gamma\backslash D = S$ realize S as a quotient of D, a homogeneous space for the action of a real algebraic group G, by the action of $\Gamma < G$, an arithmetic subgroup. Let $S' \subseteq S$ be a special subvariety of S realized as $\pi(D')$ for $D' \subseteq D$ a homogeneous space for an algebraic subgroup of G. Let $X \subseteq S$ be an irreducible subvariety of S not contained in any proper weakly special subvariety of S. Assume that the intersection of X with S' is persistently likely meaning that whenever $\zeta : S_1 \to S$ and $\xi : S_1 \to S_2$ are maps of Shimura varieties (meaning regular maps of varieties induced by maps of the corresponding Shimura data) with ζ finite, $\dim \xi\zeta^{-1}X + \dim \xi\zeta^{-1}S' \geq \dim \xi S_1$. Then $X \cap \bigcup_{g\in G, \pi(gD') \text{ is special}} \pi(dD')$ is dense in X for the Euclidean topology.

1. Introduction

If X is an complex manifold and $f : Y \to X$ and $g : Z \to X$ are two maps from complex analytic spaces to X, then we say that an intersection between Y and Z is *unlikely* if $\dim f(Y) + \dim g(Z) < \dim X$ (where the dimension of the image may be computed as the dimension of the interior as a complex manifold) and that it is *likely* otherwise. Conjectures of Zilber-Pink type predict that in many cases of interest there are very few unlikely intersections. For example, the Zilber-Pink conjecture for Shimura varieties takes the following form. We let X be a Shimura variety and $Y \subseteq X$ a Hodge generic subvariety, meaning that there is no proper weakly special subvariety (in the sense of Shimura varieties) $T \subsetneq X$ with $Y \subseteq T$. The conclusion of the conjecture in this case is that the union of all unlikely intersections between Y and special subvarieties of X is not Zariski dense in Y. In this paper, we address the complementary question of describing the likely intersections.

This is not the first time that the likely intersection problem has been addressed. In Section 4 we discuss several specializations of our main theorem and the relation between our results and those which appear in the literature. Two notable instances are the "all or nothing" theorem of Baldi, Klingler, and Ullmo [7] on the density of the typical Hodge locus and the work of Gao [11, 13] (following on the work of André, Corvaja, and Zannier [1]) on the generic rank of the Betti map, from which a sufficient condition for the density of torsion in subvarieties of abelian schemes is derived.

We formulate and prove our main theorem (Theorem 3.4) in terms of analytic maps from complex algebraic varieties to definable complex quotient spaces. See Section 2 for details. In brief, a definable complex quotient space S is a complex analytic space which may be presented as a double coset space $\Gamma\backslash G/M$ where G is the connected component of the real points of an algebraic group, $M \leq G$ is a suitable subgroup, and $\Gamma \leq G$ is a discrete subgroup together with a choice of a definable (in a fixed o-minimal expansion of the real numbers) fundamental set $\mathcal{F}$. Examples of such definable complex quotient spaces include complex tori, Shimura varieties, Hopf manifolds, and (mixed) period spaces. We then define a special subvariety of a definable complex quotient space S to be the image of a map of definable complex quotient space $S' \to S$ (or possibly taken only from some subcategory of definable complex quotient spaces) which is a definable map induced by an algebraic group homomorphism followed by translation. For example, the special subvarieties of complex tori would be translates by torsion points of subtori and the special subvarieties of period spaces would come from period subdomains. We must also consider a more general class of weakly special varieties, which are the fibers of maps of definable complex quotient spaces and the images of these fibers under other maps of definable complex quotient spaces. We identify a condition we call *well-parameterization of weakly specials* whereby all of the weakly special varieties in some given definable complex quotient space come from those appearing in countably many families of weakly specials. In our applications this condition is easy to verify as we restrict to subcategories of definable complex quotient spaces for which there are only countably many morphisms all told.

Consider $S = \Gamma\backslash G/M = \Gamma\backslash D$ and $D' \subseteq D$ a homogeneous space for a subgroup $G' \leq G$. We will write $\pi_\Gamma : D \to S$ for the quotient map. It may happen that $\pi_\Gamma(gD') \subseteq S$ is a special subvariety of S for many choices of $g \in G$, where "many" might mean that the set of such g is dense in G. For example, if $D = \mathfrak{h}^2$ is the Cartesian square of the upper half plane, $\Gamma = \mathrm{PGL}_2^2(\mathbb{Z})$ and $D' = \{(\tau, \tau) : \tau \in \mathfrak{h}\}$ is the diagonal, then all modular plane curves may be expressed as $\pi_\Gamma(gD')$ as g ranges through $\mathrm{PGL}_2^2(\mathbb{Q})$ acting via pairs of rational linear transformations. In such a situation we might expect that if X is a quasiprojective complex algebraic variety and $f : X^{\mathrm{an}} \to S$ is a definable complex analytic map from the analytification of X to S, then the set $f^{-1} \bigcup_{g \in G, \pi_\Gamma(gD') \text{ special}} \pi_\Gamma(gD')$ of special intersections is dense in X provided that the intersections are likely in the sense that $\dim f(X) + \dim D' \geq \dim S$. This is not quite right as the intersection may become unlikely after transformation through a special correspondence. We account for this complication with the notion of *persistently likely intersections*. Our main Theorem 3.4 asserts that if we know that the Ax-Schanuel theorem holds for our given category of definable complex quotient spaces (which satisfies some natural closure properties and the well-parameterization of weakly specials condition), then, in fact, when the intersection of $f(X)$ with $\pi_\Gamma(D')$ is persistently likely, the set of special intersections is dense in X.

Unsurprisingly, Ax-Schanuel theorems play key roles in the existing proofs of the density of special intersections. What may be surprising is that our argument is not an abstraction of the proofs appearing, for example, in [7] or [11]. Instead, our arguments are inspired by the work of Aslanyan and Kirby [3], especially with the proof of their Theorem 3.1. The reader will recognize a resemblance between our notion of persistent likeliness and the J-broad and J-free conditions of [3], which themselves extend freeness and rotundity conditions from earlier works on existential closedness as a converse to Schanuel-type statements. While the contents of our arguments differ, the structure of many of the results of Daw and Ren in [10] inspired our approach.

Our own interest in the likely intersections problem was motivated by [21, Conjecture 3.13] in the second author's work with Pila on effective versions of the Zilber-Pink conjecture.

This paper is organized as follows. In Section 2 we define definable quotient spaces and develop some of their basic theory. These definitions and results owe their form to the formalism of Bakker, Brunebarbe, Klingler and Tsimerman [5, 4] used to study arithmetic quotients and more generally mixed period spaces. In that section we express precisely what the Ax-Schanuel condition means and show that under well-parameterization of weakly specials hypothesis it implies a uniform version of itself. Section 3 is devoted to the statement and proof of our main theorem on dense special intersections. In Section 4 we detail several specializations of the main theorem, including to Hodge loci, intersections with modular varieties, and density of torsion in subvarieties of abelian schemes.

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2. Complex quotient spaces and S-special varieties

We express our theorems on likely intersections in terms of classes of definable complex quotient spaces. Our formalism is similar to what appears in [5], though we explicitly include the fundamental domain giving the definable structure as part of our data.

Throughout we work in an appropriate o-minimal expansion of the real field $\mathbb{R}$ (usually $\mathbb{R}_{\mathrm{an,exp}}$), and the word *definable* is meant with respect to this choice of o-minimal structure.

Definition 2.1. A *definable quotient space* is given by the data of

- a definable group G,
- a definable compact subgroup $M \leq G$ of G,
- a discrete subgroup $\Gamma \leq G$ of G, and
- $\mathcal{F} \subseteq D := G/M$ a definable open fundamental set for the action of Γ on D (that is, $D = \bigcup_{\gamma \in \Gamma} \gamma\mathcal{F}$ and there is a finite subset $\Gamma' \subseteq \Gamma$ so that if $x \in \mathcal{F}$ and $\gamma x \in \mathcal{F}$ for some $\gamma \in \Gamma$, then $\gamma \in \Gamma'$) for which the closure $\overline{\mathcal{F}}$ of $\mathcal{F}$ is contained in $\bigcup_{\gamma \in \Gamma''} \gamma\mathcal{F}$ for some finite subset $\Gamma'' \subseteq \Gamma$.

We write $S_{\Gamma,G,M;\mathcal{F}}$ both for the quotient space $\Gamma\backslash D = \Gamma\backslash G/M$ regarded as a definable, real analytic space where the definable structure comes from $\mathcal{F}$, and for the data $(G, M, \Gamma, \mathcal{F})$ giving this space. We denote the corresponding quotient map by $\pi_\Gamma : D \to S_{\Gamma,G,M;\mathcal{F}}$. When the data are understood, we suppress them and write S for $S_{\Gamma,G,M;\mathcal{F}}$ and $\pi : D \to S$ for the quotient map.

The class of definable quotient spaces forms a category DQS with the following notion of a morphism.

Definition 2.2. A *morphism* $f : S_{\Gamma_1,G_1,M_1;\mathcal{F}_1} \to S_{\Gamma_2,G_2,M_2;\mathcal{F}_2}$ is given by a definable map of groups $\varphi : G_1 \to G_2$ and an element $a \in G_2$ for which

- $\varphi(M_1) \leq M_2$,
- $\varphi(\Gamma_1) \leq a^{-1}\Gamma_2 a$, and
- there is a finite set $\Xi \subseteq \Gamma_2$ with $a\overline{\varphi}(\mathcal{F}_1) \subseteq \bigcup_{\xi\in\Xi} \xi\mathcal{F}_2$, where $\overline{\varphi} : D_1 \to D_2$ is the map induced on the quotient space.

The induced map on the double quotient space is a definable real analytic map which we also denote by f.

The category DQS has a terminal object and is closed under fiber products.

Proposition 2.3. *There is a terminal object in* DQS.

Proof. Take $G = M = \Gamma = \{1\}$ to be trivial group and then let $\mathcal{F} := G/M$, which is a singleton. The one point space $\{*\} = S_{G,\Gamma,M;\mathcal{F}}$ is a definable quotient space and for any definable quotient space $S = S_{G',\Gamma',M';\mathcal{F}'}$, the unique set theoretic map $S \to \{*\}$ is induced by the unique map of groups $\varphi : G' \to \{1\}$ and $a = 1 \in \{1\}$. $\square$

Proposition 2.4. *Given maps $f : S_2 \to S_1$ and $g : S_3 \to S_1$ of definable quotient spaces, there are maps of definable quotient spaces $\overline{g} : S_4 \to S_2$ and $\overline{f} : S_4 \to S_3$ fitting into the following Cartesian square.*

$$\begin{tikzcd} S_4 \arrow[r, "\overline{f}"] \arrow[d, "\overline{g}"'] & S_3 \arrow[d, "g"] \\ S_2 \arrow[r, "f"] & S_1 \end{tikzcd}$$

Proof. Express $S_i = S_{G_i,\Gamma_i,M_i;\mathcal{F}_i}$ and $f : S_2 \to S_1$ as $\Gamma_2 x M_2 \mapsto \Gamma_1 a_2 \varphi(x) M_1$ and $g : S_3 \to S_1$ as $\Gamma_3 x M_3 \mapsto \Gamma_1 a_3 \psi(x) M_1$ where $\varphi : G_2 \to G_1$ and $\psi : G_3 \to G_1$ are definable group homomorphisms, $a_2 \in G_1$ and $a_3 \in G_1$. Let $G_4 := G_2 \times_{G_1} G_3$, $M_4 := M_2 \times_{M_1} M_3$, and $\Gamma_4 := \Gamma_2 \times_{\Gamma_1} \Gamma_3$. For the fiber products defining G_4 and M_4 we use the maps $\varphi : G_2 \to G_1$ (and its restriction to M_2) and $\psi : G_3 \to G_1$ (and its restriction to M_3). In defining Γ_4, we use the maps $x \mapsto a_2\varphi(x)a_2^{-1}$ and $x \mapsto a_3\psi(x)a_3^{-1}$. Regarding G_4 as a subgroup of $G_2 \times G_3$ and then $D_4 := G_4/M_4$ as a subset of $G_2/M_2 \times G_3/M_3 = D_2 \times D_3$, we let $\mathcal{F}_4 := D_4 \cap (\mathcal{F}_2 \times \mathcal{F}_3)$. $\square$

A useful observation is that morphisms of definable quotient spaces always factor as a surjective map followed by a map induced by an inclusion of subgroups.

Proposition 2.5. *Every map $f : S_1 \to S_2$ of definable quotient spaces fits into a commutative diagram*

$$\begin{tikzcd} S_1 \arrow[rr, "f"] \arrow[dr, "q"'] & & S_2 \\ & S_3 \arrow[ur, "p"'] & \end{tikzcd}$$

where $q : S_1 \to S_3$ is surjective and $p : S_3 \to S_2$ is induced by an inclusion of subgroups. If we further assume that the group M_2 in the presentation of $S_2 = S_{G_2,\Gamma_2,M_2;\mathcal{F}}$ is compact, then p has compact fibers.

Proof. Take $S_i = S_{G_i,\Gamma_i,M_i;\mathcal{F}_i}$ for $1 \leq i \leq 2$ and let $\varphi : G_1 \to G_2$ be a definable homomorphism and $a \in G_2$ so that $f : S_1 \to S_2$ is given by $\Gamma_1 x M_1 \mapsto \Gamma_2 a\varphi(x) M_2$.

Define $G_3 := \varphi(G_1) \leq G_2$, a definable group. Set $\Gamma_3 := \varphi(\Gamma_1)$, $M_3 := \varphi(M_1)$, and $\mathcal{F}_3 := \overline{\varphi}(\mathcal{F}_1)$ where $\overline{\varphi} : G_1/M_1 \to G_3/M_3$ is the induced map. Since $\overline{\varphi}$ is a definable open map, $\mathcal{F}_3$ is open and clearly $\mathcal{F}_3$ is a fundamental set of the action of Γ_3 on G_3/M_3. Set $S_3 := S_{G_3,\Gamma_3,M_3;\mathcal{F}}$ and let $q : S_1 \to S_3$ be the map $\Gamma_1 x M_1 \mapsto \Gamma_3 \varphi(x) M_3$. The map $p : S_3 \to S_2$ is then given by $\Gamma_3 x M_3 \mapsto \Gamma_2 a x M_2$. The fibers of p are contained in a finite union of sets definably homeomorphic to the homogeneous space $(M_2 \cap G_3)/M_3$, which is compact if M_2 is. $\square$

For the problems we consider in this paper we require that our definable quotient spaces come equipped with a complex structure and for the domain D to arise as a subset of an algebraic variety.

Definition 2.6. A *definable complex quotient space* is a definable quotient space $S_{\Gamma,G,M;\mathcal{F}}$ together with the data of a real algebraic group $\mathbf{G}$ and an algebraic subgroup $\mathbf{B} \leq \mathbf{G}_{\mathbb{C}}$ of the base change of $\mathbf{G}$ to $\mathbb{C}$ for which $G = \mathbf{G}(\mathbb{R})^+$ is the connected component of the identity in the real points of a real algebraic group $\mathbf{G}$, $M = \mathbf{B}(\mathbb{C}) \cap G$, and $D = G/M \subseteq (\mathbf{G}/\mathbf{B})(\mathbb{C}) =: \check{D}(\mathbb{C})$ is an open domain in the complex points of the algebraic variety $\check{D}$.

A morphism $f : S_{\Gamma_1,G_1,M_1;\mathcal{F}_1} \to S_{\Gamma_2,G_2,M_2;\mathcal{F}_2}$ of definable complex quotient spaces is a morphism of definable quotient spaces for which the definable map of groups is given by a map of algebraic groups $\varphi : \mathbf{G}_1 \to \mathbf{G}_2$ for which $\varphi(\mathbf{B}_1) \leq \mathbf{B}_2$.

The class of definable complex quotient spaces with this notion of morphism forms a category $\mathsf{D\mathbb{C}QS}$.

We leave it to the reader to check that the proofs of the basic closure properties for the category DQS, such as the existence of a terminal object and closure under fiber products, go through for the category $\mathsf{D\mathbb{C}QS}$. In practice, the morphisms in $\mathsf{D\mathbb{C}QS}$ we consider satisfy a stronger conclusion than what Proposition 2.5 gives. That is, in practice, a map $f : S_{G_1,\Gamma_1,M_1;\mathcal{F}_1} \to S_{G_2,\Gamma_2,M_2;\mathcal{F}_2}$ in $\mathsf{D\mathbb{C}QS}$ is given by a map of algebraic groups $\varphi : \mathbf{G}_1 \to \mathbf{G}_2$ (and an element $a \in G_2$) for which M_1 is a finite index subgroup of $\varphi^{-1}(M_2)$. It then follows from the proof of Proposition 2.5 that f factors as $f = p \circ q$ where $q : S_1 \to S_3$ is a surjective $\mathsf{D\mathbb{C}QS}$ morphism and $p : S_3 \to S_2$ is a $\mathsf{D\mathbb{C}QS}$ morphism with finite fibers.

As we have defined definable complex quotient spaces, such a space $S_{G,\Gamma,M;\mathcal{F}}$ may have singularities. When we restrict to the case that the group M is compact, then the singularities are at worst locally isomorphic to those coming from a quotient by a finite group. In our applications, we will consider only cases where these quotients may be desingularized by passing to a finite cover by another definable complex quotient space.

For some purposes we may wish to restrict to an even smaller category S. We always assume about our given category S of definable complex quotient spaces that it satisfies some basic closure properties. Let us specify these with the following convention.

Convention 2.7. *The category* S *is a subcategory of* $\mathsf{D\mathbb{C}QS}$ *satisfying the following conditions.*

- *The one point space* $\{*\}$ *is a terminal object of* S.
- *The category* S *is closed under fiber products.*

- *Every* S*-morphism* $f : S_1 \to S_2$ *factors as* $f = p \circ q$ *where* $q : S_1 \to S_3$ *is a surjective* S*-morphism and* $p : S_3 \to S_2$ *is an* S*-morphism with finite fibers.*

In Section 3 we will impose an additional restriction on S.

Definition 2.8. If $f : S_1 \to S_2$ is an S-morphism, then the image $f(S_1)$ is called an S-*special* subvariety of S_2.

In Definition 2.8 we refer to $f(S_1)$ as a special *subvariety.* It is, in fact, always a complex analytic subvariety of S_2. Indeed, if we factor $f = p \circ q$ as given by Convention 2.7, then $p : S_3 \to S_2$ is a finite, and hence, proper, map of complex analytic spaces so that its image $p(S_3) = f(S_1)$ is a complex analytic subvariety of S_2. Using this observation, we may modify Definition 2.8 to require the morphism $f : S_1 \to S_2$ witnessing that $f(S_1)$ is an S-special subvariety of S_2 to be finite.

Definition 2.9. An S-*family of weakly special subvarieties of* $S \in \mathsf{S}$ is given by a pair of S-morphisms

$$\begin{tikzcd} & S_1 \arrow[dl, "\zeta"'] \arrow[dr, "\xi"] & \\ S & & S_2 \end{tikzcd}$$

for which ζ is a finite map over its image and ξ is surjective. For each $b \in S_2$, the image $\zeta\xi^{-1}\{b\}$ is a *weakly* S*-special* subvariety of S.

Remark 2.10. An S-special variety is weakly S-special as if $f : S_1 \to S$ expresses $f(S_1)$ as an S-special subvariety of S with f finite, then we can take $\zeta = f$, $S_2 = \{*\}$, $\xi : S_1 \to S_2$ the unique map to $\{*\}$. If we take S to be the category $\mathsf{D}\mathbb{C}\mathsf{QS}$, then the converse that every weakly S-special variety is actually S-special holds.

In another extreme, every singleton in $S \in \mathsf{S}$ is a weakly special variety witnessed by $S_1 = S_2$ and $\xi = \zeta = \mathrm{id}_S$.

For our results it will be important that all S-weakly special varieties come from countably many S-families of weakly specials. We isolate it as an hypothesis and verify it in cases of interest.

Definition 2.11. We say that the weakly S-specials of $S \in \mathsf{S}$ are *well-parameterized* if there are countably many S-families of weakly special subvarieties of S,

$$\begin{tikzcd} & S_{1,i} \arrow[dl, "\zeta_i"'] \arrow[dr, "\xi_i"] & \\ S & & S_{2,i} \end{tikzcd}$$

for $i \in \mathbb{N}$ so that for every weakly S-special subvariety $S' \subseteq S$ of S there is some $i \in \mathbb{N}$ and $c \in S_{2,i}$ so that $S' = \zeta_i\xi_i^{-1}\{c\}$. More generally, we say that the weakly S-specials are well-parameterized if for every $S \in \mathsf{S}$ the weakly S-specials of S are well-parameterized.

Remark 2.12. When S is itself countable, by which we mean that there are only countably many objects in S and the set of S-morphisms between any two such objects is itself countable, then the weakly S-specials are well-parameterized.

Remark 2.13. We are not sure whether the well-parameterization condition always holds. From the theory of Douady spaces, we know that all complex analytic subvarieties of a given definable complex quotient space S are parameterized by countably many complex analytic families of analytic spaces. However, we do not see why when parameterizing weakly special varieties, the Douady universal family should arise as weakly special family as defined in Definition 2.9. The Hopf manifold construction may be instructive here.

Let us indicate now the key functional transcendence condition which may hold in a category S of definable complex quotient spaces.

Definition 2.14. We say that the category S satisfies the *Ax-Schanuel condition* if whenever $f : X^{\mathrm{an}} \to S' \subseteq S_{\Gamma,G,M;\mathcal{F}} \in \mathsf{S}$ is a definable complex analytic map from the analytification of a complex algebraic variety X to a weakly S-special variety $S' \subseteq S$, $k \in \mathbb{Z}_+$ is a positive integer, and $(\gamma, \widetilde{\gamma}) : \Delta^k \to X \times D$ is a complex analytic map, where

$$\Delta = \{z \in \mathbb{C} : \|z\| < 1\} ,$$

with $\pi_\Gamma \circ \widetilde{\gamma} = f \circ \gamma$, then either

$$\operatorname{tr.deg}_{\mathbb{C}}(\mathbb{C}(\gamma, \widetilde{\gamma})) \geq \dim S' + \operatorname{rk}(d\widetilde{\gamma})$$

or $f(\gamma(\Delta^k))$ is contained in a proper weakly S-special subvariety of S'.

Under the hypothesis that the weakly S-specials are well-parameterized, the Ax-Schanuel condition implies a uniform version of itself.

Since our statement of this uniform version, expressed as Proposition 2.19 below, is a bit dense, we take this opportunity to explain it with a few words. Basically, what it says is that if we are given an S-family of weakly special varieties and a family of algebraic varieties which might witness the failure of the transcendence degree lower bound in the Ax-Schanuel property, then weakly special variety in the alternative provided by the Ax-Schanuel property may be chosen from one of finitely many preassigned S-families of weakly special varieties.

Before proving Proposition 2.19 we require two lemmas. The first describes families of weakly specials algebraically. The second allows us to recast Ax-Schanuel in differential algebraic terms.

Lemma 2.15. *Let $f : X^{an} \to S_{\Gamma,G,M;\mathcal{F}} =: S \in \mathsf{S}$ be a definable complex analytic map from a complex algebraic variety to a definable complex quotient space in S. Let*

$$\begin{tikzcd} & S_1 \arrow[dl, "\zeta"'] \arrow[dr, "\xi"] & \\ S & & S_2 \end{tikzcd}$$

be an S-family of weakly specials. Then there are algebraically constructible sets B and $T \subseteq X \times B$ so that the set of fibers $\{T_b : b \in B(\mathbb{C})\}$ is equal to $\{f^{-1}\zeta\xi^{-1}\{c\} : c \in S_2\}$.

Proof. By the Riemann Existence Theorem [22, Théorème 5.2], there is an algebraic variety X', a regular map of algebraic varieties $\zeta' : X' \to X$ and an analytic map $f' : (X')^{\mathrm{an}} \to S_1$ realizing $(X')^{\mathrm{an}}$ as the fiber product $X \times_S S_1$. The fiber equivalence relation

$$E_{\xi \circ f'} := \{(x, y) \in X' \times X' : \xi(f'(x)) = \xi(f'(y))\}$$

is analytic and definable, and hence algebraic by the definable Chow theorem [20]. The quotient $B := X'/E_{\xi \circ f'}$ may be realized as a constructible set. Let us write $\nu : X' \to B$ for the quotient map. We may then take

$$T := \{(x, b) \in X \times B : (\exists x' \in X)\zeta'(x') = x \text{ and } \nu(x') = b\} .$$

□

Definition 2.16. Let $f : X^{\mathrm{an}} \to S \in \mathsf{S}$ be a definable complex analytic map from the complex algebraic variety X to the definable complex quotient space S in S. We say that a subvariety $Y \subseteq X$ is *relatively weakly* S*-special of relative dimension at most* d if there is a weakly S-special $S' \subseteq S$ of dimension at most d for which $Y = f^{-1}S'$.

Note that in Definition 2.16, because we allow for the possibility that f is not a finite map, it could happen that the dimension of Y itself is greater than d. On the other hand, the intersection of S' with $f(X)$ may even be empty! Thus, the dimension of Y could be less than d.

It follows from Lemma 2.15 that if the weakly S-specials are well-parameterized, then for any definable complex analytic map $f : X^{\mathrm{an}} \to S \in \mathsf{S}$ from a complex algebraic variety X to some definable complex quotient space S in S that we can recognize the pullbacks under f of prespecials in the sense that for each number d the collection of relatively weakly S-special subvarieties of dimension at most d comprise a countable collection of algebraic families of subvarieties of X.

Definition 2.17. Let $f : X^{\mathrm{an}} \to S \in \mathsf{S}$ be a definable complex analytic map from the complex algebraic variety X to to a definable complex quotient space S in S. Given any field M over which X and a countable collection of families of relatively weakly S-special subvarieties of X including all such relatively weakly S-special subvarieties are defined, by an M*-relatively weakly* S*-special variety of dimension at most* d we mean an M-variety of the form Y_b where $Y \subseteq X \times B$ is an algebraic family of weakly S-specials of dimension at most d and $b \in B(M)$.

Using the Seidenberg embedding theorem we may reformulate the Ax-Schanuel property in differential algebraic terms. To be completely honest, the embedding theorem as stated and proven by Seidenberg in [24, 25] is not quite sufficient in that he starts with a finitely generated differential subfield $K \subseteq \mathcal{M}(U)$ of a differential field of meromorphic functions on the open domain $U \subseteq \mathbb{C}^n$ and then shows that for any finitely generated differential field extension L of K at the cost of shrinking U to some open subdomain $V \subseteq U$ we may embed L into $\mathcal{M}(V)$ over the embedding of K. For our purposes, we will need to start with a possibly countably generated differential field $K \subseteq \mathcal{M}(U)$. The necessary extension of embeddings theorem is a consequence of the the Cauchy-Kovalevskaya theorem [26] and appears as Theorem 1 of [19]. Iterating this construction countably many steps, we see that if $K \subseteq \mathcal{M}(U)$ is a countable differential subfield of the meromorphic functions on some open domain in $\mathbb{C}^n$ and L is a countably generated differential field extension of K, then L embeds into the differential field of germs of meromorphic functions at some point $x \in U$ over the embedding of K.

We recall the generalized Schwartzian and generalized logarithmic derivative constructions from [23]. Consider $S = S_{G,\Gamma,M;\mathcal{F}}$ a definable complex quotient space and fix an integer k. From the action $\mathbf{G} \curvearrowright \check{D}$ of the algebraic group $\mathbf{G}$ on the quasiprojective algebraic variety $\check{D}$ and a positive integer k, there is a differentially constructible map $\widetilde{\chi} : \check{D} \to Z$ from $\check{D}$ to some algebraic variety Z so that for any differential field $(L, \partial_1, \ldots, \partial_k)$ extending $\mathbb{C}$ (where the derivations ∂_i commute and vanish on $\mathbb{C}$) we have that for $x, y \in \check{D}(L)$,

$$\widetilde{\chi}(x) = \widetilde{\chi}(y) \Longleftrightarrow (\exists g \in \mathbf{G}(C))gx = y$$

where

$$C = \{a \in L : \partial_i(a) = 0 \text{ for } 1 \leq i \leq k\}$$

is the common constant field of L. In particular, if this common constant field is $\mathbb{C}$, then we may express the quotient of $\check{D}$ by the $\mathbf{G}(\mathbb{C})$ as the image of $\widetilde{\chi}$. When $f : X^{\mathrm{an}} \to S$ is a definable, complex analytic map from the analytification of a quasiprojective algebraic variety X to S, then we may define a differentially analytically constructible function $\chi : X \to Z$ by the rule that for any meromorphic $\gamma : U \to X$ (where $U \subseteq \mathbb{C}^k$ is an open domain in $\mathbb{C}^k$), $\chi(\gamma) := \widetilde{\chi}(\pi_\Gamma^{-1}(f(\gamma))$ where π_Γ^{-1} is any branch of the inverse of π_Γ. Theorem 3.12 of [23] shows that χ is actually differentially constructible. (That theorem is stated in the case where $X^{\mathrm{an}} = S$ and $f = \mathrm{id}_S$, but the proof goes through in the more general case.)

Lemma 2.18. *Suppose that* S *satisfies the Ax-Schanuel condition and that the weakly* S*-specials are well-parameterized. Let* $f : X^{an} \to S_{\Gamma,G,M;\mathcal{F}} =: S \in \mathsf{S}$ *be a definable complex analytic map from a complex algebraic variety to a definable complex quotient space in* S*. Let* K *be a countable subfield of* $\mathbb{C}$ *over which* X *and a complete collection of algebraic families of relatively weakly* S*-special varieties are defined. Let* k *and* d *be two positive integers. Then for any differential field* $(L, \delta_1, \ldots, \delta_k)$ *with* k*-commuting derivations for which* K *is a subfield of the constants* C *of* L *and* C *is algebraically closed and any* C*-relatively weakly* S*-special* Y *of dimension at most* d*, if* $(\gamma, \widetilde{\gamma}) \in Y(L) \times \check{D}(L)$ *satisfies* $\chi(\gamma) = \widetilde{\chi}(\widetilde{\gamma})$*,* $\mathrm{rk}((\delta_i\gamma)_{1 \leq i \leq k}) = k$ *and* $\mathrm{tr.deg}_C\, C(\gamma, \widetilde{\gamma}) < d + k$*, then there is a* C*-relatively weakly* S*-special* $Z \subsetneq Y$ *for which* $\gamma \in Z(L)$*.*

Proof. Consider Y and $(\gamma, \widetilde{\gamma}) \in Y(L)$ as in the statement of the lemma. Let M be a countable differential subfield of L with an algebraically closed field of constants C' containing K and over which Y and the point $(\gamma, \widetilde{\gamma})$ are defined. By the embedding theorem, we may realize M as a differential field of germs of meromorphic functions. Let $S' \subseteq S$ be the S-weakly special variety of dimension at most d for which $Y = f^{-1}S'$. By Ax-Schanuel applied to γ and $\widetilde{\gamma}$ regarded as meromorphic functions, there is a proper weakly S-special subvariety $S'' \subsetneq S'$ with the image of $f \circ \gamma$ contained in S''. The algebraic variety $Z := f^{-1}S''$ is then relatively $\mathbb{C}$-weakly special. Let $M' := M^{\mathrm{dc}}$ be the differential closure of M and $M'' := M(\mathbb{C})^{\mathrm{dc}}$ be the differential closure of the differential field generated over M by $\mathbb{C}$. In M'', γ satisfies the condition that it belongs to a C-relatively weakly special variety of relative dimension strictly less than d where C is the constant field. As M'' is an elementary extension of M', the same is true in M'. Since the constant field of the differential closure is the algebraic of the constant field of the initial field, we see that γ belongs to a C-relatively weakly special variety of relative dimension strictly less than d. □

A uniform version of the Ax-Schanuel condition follows from Lemma 2.15 using the compactness theorem.

Proposition 2.19. *Suppose that* S *satisfies the Ax-Schanuel condition and that the weakly* S*-specials are well-parameterized. Let* $f : X^{an} \to S = S_{\Gamma,G,M;\mathcal{F}} \in \mathsf{S}$

be definable complex analytic map from the analytification of a complex algebraic variety X to a definable complex quotient space S in S*. Given an* S*-family*

$$\begin{tikzcd} & S_1 \arrow[dl, "\zeta"'] \arrow[dr, "\xi"] & \\ S & & S_2 \end{tikzcd}$$

of weakly S*-special subvarieties of S, a positive integer $k \in \mathbb{Z}_+$, and a family $Y \subseteq (X \times \check{D}) \times B$ of subvarieties of $X \times \check{D}$, then there are finitely many* S*-families of weakly specials*

$$\begin{tikzcd} & S_{1,i} \arrow[dl, "\zeta_i"'] \arrow[dr, "\xi_i"] & \\ S & & S_{2,i} \end{tikzcd}$$

for $1 \leq i \leq n$ so that for any pair of parameters $b \in B$ and $c \in S_2$ and analytic map $(\gamma, \widetilde{\gamma}) : \Delta^k \to Y_b \subseteq X \times D$ with $\gamma(\Delta^k) \subseteq \zeta\xi^{-1}\{c\} =: S'_c$, $f \circ \gamma = \pi_\Gamma \circ \widetilde{\gamma}$, $\mathrm{rk}(d\widetilde{\gamma}) = k$, and $\dim Y_b < k + \dim S'_c$, there is some $i \leq n$ and $d \in S_{2,i}$ for which $f \circ \gamma(\Delta^k) \subseteq \zeta_i\xi_i^{-1}\{d\} \subsetneq S'_c$.

Proof. Apply the compactness theorem to Lemma 2.15. See the proofs of [2, Theorem 3.5] or [16, Theorem 4.3] for details on how to formalize the compactness argument. □

3. Density of special intersections

In this section we state and prove our general theorem that, when persistently likely, intersections with special varieties are dense.

Throughout this section S is a category of definable complex quotient spaces satisfying our usual hypotheses from Convention 2.7, and then two more basic properties. Let us specify these with the following convention.

Convention 3.1. *The category* S *of definable complex quotient spaces satisfies the following conditions.*

- S *is closed under fiber products.*
- *The terminal definable complex quotient (one point) space $\{*\}$ belongs to* S *as do the unique maps $S \to \{*\}$ for $S \in \mathsf{S}$.*
- *If $f : S_1 \to S_2$ is an* S*-morphism then there are* S*-morphisms $q : S_1 \to S_3$ and $p : S_3 \to S_2$ so that $f = p \circ q$, q is surjective, and p has finite fibers.*
- *For every $S \in \mathsf{S}$ there is some smooth $S' \in \mathsf{S}$ and a finite surjective* S*-morphism $S' \to S$.*
- *Ax-Schanuel holds in* S*.*

With the following definition we specify what is meant by intersections being persistently likely.

Definition 3.2. Let $S = S_{G,\Gamma,M;\mathcal{F}} \in \mathsf{S}$. We suppose that the inclusion $S' = S_{G',\Gamma\cap G',M\cap H;\mathcal{F}'} \hookrightarrow S$ is an S-morphism where if $D = G/M$ and $D' = G'/(G' \cap M)$, then $\mathcal{F}' = \mathcal{F} \cap D'$. Suppose that $G' \leq G$ and $D' = G'/(M \cap G')$. Let $f : X^{\mathrm{an}} \to S$ be a definable complex analytic map. We say that X and S' have *likely intersection* if $\dim f(X) + \dim S' \geq \dim S$, where here, the dimension is the o-minimal dimension. We say the intersection is *persistently likely* if whenever $\zeta : S_1 \to S$ and $\xi : S_1 \to S_2$ are S-morphisms with ζ finite and ξ surjective, then $\dim \xi(\zeta^{-1} f(X)) + \dim \xi(\zeta^{-1} S') \geq \dim S_2$.

Note that this definition does not actually require $f(X) \cap S'$ to be non-empty for the intersection to be likely.

Remark 3.3. In Definition 3.2, we have consider only one special subvariety S' of S, but we really intend to consider many. With the notation as in Definition 3.2, for any $g \in G$, the set $\pi_\Gamma(gD' \cap \mathcal{F}) \subseteq S$ is locally (away from $\pi_\Gamma(gD' \cap \partial(\mathcal{F}))$) a complex analytic subvariety of S of dimension equal to that of D'. Indeed, in several cases of interest for many choices of g (where this may mean that the set of suitable g is dense in G) the set $\pi_\Gamma(gD')$ is actually a special subvariety of S. Persistent likelihood of the intersection of X with S' is equivalent to the persistent likelihood of the intersection of X with these $\pi_\Gamma(gD')$.

With this definition in place we may now state our main theorem.

Theorem 3.4. *Let $S = S_{G,\Gamma,M;\mathcal{F}} \in \mathsf{S}$ and let $f : X^{an} \to S$ be a definable complex analytic map from the irreducible quasi-projective complex algebraic variety X to S. Suppose that $S' \subseteq S$ is a special subvariety expressible as $\pi_\Gamma(D')$ where $D' \subseteq D = G/M$ is a homogeneous space in D. Suppose moreover that the intersection of X with S' is persistently likely. Let $U \subseteq X(\mathbb{C})$ be an open subset of the complex points of X. Then the set $B := \{g \in G : f(U) \cap \pi_\Gamma(gD' \cap \mathcal{F}) \neq \varnothing\}$ has nontrivial interior. In particular, if the set $\{g \in G : \pi_\Gamma(gD')$ is a special subvariery of $S\}$ is Euclidean dense in G, then the set of special intersections,*

$$f^{-1} \bigcup_{g \in G, \pi_\Gamma(gD') \text{ is special}} \pi_\Gamma(gD')$$

is dense in X for the Euclidean topology.

Proof. We break the proof of Theorem 3.4 into several claims. The claims at the beginning of the proof are really just reductions permitting us to consider a simpler situation. The main steps of the proof begin with Claim 3.4.9 in which we compute the dimension of the incidence correspondence R. We then use this computation to show that B and G have the same o-minimal dimension, so that B has nontrivial interior in G.

Claim 3.4.1. *We may assume that $f : X^{an} \to S$ is an embedding.*

Proof of Claim: The equivalence relation $E_f := \{(x, y) \in X \times X : f(x) = f(y)\}$ is a definable and complex analytic subset of the quasi-projective algebraic variety $X \times X$. Hence, by the definable Chow theorem, E_f is itself algebraic. Let Y be a nonempty Zariski open subset of X/E_f, considered as constructible set. Then f induces an embedding $\overline{f} : Y \hookrightarrow S$ for which the image of $\overline{f}$ is dense in $f(X)$. Shrinking U, we may assume that $f(U) \subseteq \overline{f}(Y)$ and then replacing U by $U' := \overline{f}^{-1} f(U) \subseteq Y(\mathbb{C})$, we see that if the theorem holds for $\overline{f} : Y^{\text{an}} \to S$ and U', then it also holds for $f : X^{\text{an}} \to S$ and U. ✠

With Claim 3.4.1 in place, from now on we will regard X as a locally closed subvariety of S. With the next claim we record the simple observations that it suffices to prove the theorem for any given open subset of U in place of U and that we take U to be definable.

Claim 3.4.2. *If Theorem 3.4 holds for some nonempty open $V \subseteq U$ in place of U, then it holds as stated. Moreover, we may assume that U is definable.*

Proof of Claim: The set $\{g \in G : V \cap \pi(gD' \cap \mathcal{F}) \neq \varnothing\}$ is a subset of $\{g \in G : U \cap \pi(gD' \cap \mathcal{F}) \neq \varnothing\}$. Hence, if the former set has nonempty interior, so does the latter. For the "moreover" clause apply the main body of the claim to the case that $V \subseteq U$ is a nonempty open ball. ✠

From now on we will take U to be definable and will continue to refer to the open subset of X under consideration as U even after taking various steps to shrink it.

Another basic reduction we shall employ is that it suffices to prove the theorem for a finite cover of S.

Claim 3.4.3. *If $\rho : \widetilde{S} \to S$ is a finite surjective S-morphism, then we may find an instance of the statement of Theorem 3.4 with $\widetilde{S}$ in place of S so that the truth of Theorem 3.4 for $\widetilde{S}$ implies the result of S.*

Proof of Claim: Filling the Cartesian square

$$\begin{tikzcd} Y \arrow[r, "\overline{f}"] \arrow[d, "\overline{\rho}"'] & \widetilde{S} \arrow[d, "\rho"] \\ X^{\mathrm{an}} \arrow[r, "f"] & S \end{tikzcd}$$

we obtain a complex analytic space $Y := X^{\mathrm{an}} \times_S \widetilde{S}$. Since $\overline{\rho} : Y \to X^{\mathrm{an}}$ is finite, Y is itself the analytification of an algebraic variety. Let $\widetilde{X}$ be a component of this algebraic variety and then let $\widetilde{U} := \overline{\rho}^{-1}U$.

The map $\rho : \widetilde{S} \to S$ comes from a homomorphism of algebraic groups $\varphi : \widetilde{\mathbf{G}} \to \mathbf{G}$ and some element $a \in G$ where $\widetilde{S} = S_{\widetilde{G},\widetilde{\Gamma},\widetilde{M};\widetilde{\mathcal{F}}}$. This map induces a map $\widehat{\rho} : \widetilde{G}/\widetilde{M} =: \widetilde{D} \to D$. Let $\widetilde{D}'$ be a component of $\widehat{\rho}^{-1}D'$. If we succeed in showing that $\{g \in G' : \overline{f}(\widetilde{U}) \cap \pi_{\widetilde{\Gamma}}(g\widetilde{D}' \cap \widetilde{\mathcal{F}}\}$ contains some nonempty open set V, then $a\varphi(V)$ would be a nonempty open subset of $\{g \in G : f(U) \cap \pi_{\Gamma}(gD' \cap \mathcal{F})\}$, as required. ✠

Let us record a useful consequence of Claim 3.4.3.

Claim 3.4.4. *We may assume that S is smooth.*

Proof of Claim: By Convention 3.1, we may find a finite and surjective S-morphism $\widetilde{S} \to S$ with $\widetilde{S}$ smooth. By Claim 3.4.3, if we know the theorem for $\widetilde{S}$, then we may deduce it for S. ✠

Another useful consequence of Claim 3.4.3 is that we may assume that $f(X)$ is dense in S with respect to the weakly special topology.

Claim 3.4.5. *We may assume that there is no proper weakly special variety $S'' \subsetneq S$ with $f(X) \subseteq S''$.*

Proof of Claim: Let us prove Theorem 3.4 by induction on the dimension of S. If $f(X) \subseteq S'' \subsetneq S$ where S'' is a weakly special variety, then we could find a finite S-morphism $\zeta : S_1 \to S$, a surjective S-morphism $\xi : S_1 \to S_2$, and a point $b \in S_2$ so that $f(X) \subseteq \zeta(\xi^{-1}\{b\})$. By Claim 3.4.3, we may assume that $S_1 = S$ and $\zeta = \mathrm{id}_S$. That is, $f(X) \subseteq \xi^{-1}\{b\} =: S_b \subsetneq S$. By Convention 3.1, the weakly special variety S_b, being the fiber product of S with the the one-point space $\{*\}$ over S_2, is definably isomorphic to a space in S. By induction on dimension, Theorem 3.4 already holds for S_b. ✠

By our hypothesis that the intersection between X and S' is persistently likely, it is, in particular, likely. Since we have reduced to the case that $f" : X^{\mathrm{an}} \to S$ is an embedding by Claim 3.4.1, we may express the likeliness of this intersection by an equation

$$\dim_{\mathbb{C}}(X) + \dim_{\mathbb{C}}(S') = \dim_{\mathbb{C}}(S) + k$$

for some nonnegative integer k. Notice that we have expressed this equality with dimensions as complex analytic spaces. Later, when we write "dim" without qualification we mean the o-minimal dimension, for which we would have

$$\dim X + \dim S' = \dim S + 2k\ .$$

By the uniform Ax-Schanuel condition, which holds in S by Convention 3.1 and Proposition 2.19, there is a finite list of families of weakly special varieties

$$\begin{tikzcd} & S_{1,i} \arrow[dl, "\zeta_i"'] \arrow[dr, "\xi_i"] & \\ S & & S_{2,i} \end{tikzcd}$$

for $1 \leq i \leq n$ where $\zeta_i : S_{1,i} \to S$ is a finite S-morphism and $\xi_i : S_{1,i} \to S_{2,i}$ is a surjective S-morphism and if ℓ is a natural number with $(\gamma, \widetilde{\gamma}) : \Delta^\ell \to U \times D$ is complex analytic with $\pi_\Gamma \circ \widetilde{\gamma} = g \circ \gamma$, $\mathrm{rk}(d\gamma) = \ell > k$, and $\widetilde{\gamma}(\Delta^\ell) \subseteq gD'$ for some $g \in G$, then for some $i \leq n$ and $b \in S_{2,i}$ we have $\widetilde{\gamma}(\Delta^\ell) \subseteq \zeta_i(\xi_i^{-1}\{b\}) \subsetneq S$.

Claim 3.4.6. *We may assume that $\zeta_i : S_{1,i} \to S$ is surjective for each $i \leq n$.*

Proof of Claim: By Claim 3.4.5, we have reduced to the case that $f(X)$ is not contained in any proper weakly special subvariety of S. For any $i \leq n$ with $\zeta_i(S_{1,i}) \neq S$, we would thus have that $\zeta_i(S_{1,i}) \cap f(X)$ is a proper complex analytic subvariety of $f(X)$. Thus, we may shrink U so that for such an i we have $\zeta_i(S_{1,i}) \cap U = \varnothing$. We will thus never encounter weakly special varieties of the form $\zeta_i(\xi^{-1}\{b\})$ with $\gamma(\Delta^\ell) \subseteq \zeta_i(\xi^{-1}\{b\}) \cap U$. Thus, we may omit these families of weakly special varieties from our list. ✠

We may adjust our family of weakly special varieties to remember only the maps $\xi_i : S_{1,i} \to S_{2,i}$.

Claim 3.4.7. *We may assume that $S_{1,i} = S$ and $\zeta_i : S_{1,i} \to S$ is the identity map $\mathrm{id}_S : S \to S$.*

Proof of Claim: Work by induction on n. In the inductive case of $n+1$, apply Claim 3.4.3 to replace S by $S_{n+1,1}$. We then need to replace $S_{i,1}$ for $i \leq n$ by $S_{i,1} \times_S S_{n+1,i}$. Conclude by induction. ✠

We shrink U once again to ensure that all of the fibers of ξ_i have the same dimension when restricted to U.

Claim 3.4.8. *We may shrink U to a smaller nonempty open set so that for all $i \leq n$ there is some number $d_i \in \mathbb{N}$ so that for all $u \in U$ we have $\dim(\xi_i^{-1}\{\xi_i(u)\} \cap f(U)) = d_i$.*

Proof of Claim: For each $i \leq n$ and each natural number $j \leq \dim U$, let

$$F_{i,j} := \{u \in U : \dim_u(f^{-1}\xi_i^{-1}\{\xi_i(f(u))\})\} = j \ .$$

Here $\dim_u(\ )$ refers to the o-minimal dimension at u.

The definable set U is the finite disjoint union of the definable sets

$$\bigcap_{i=1}^{n} F_{i,d_i}$$

as $(d_1, \ldots, d_n)$ ranges through $[0, \dim(U)]^n$. We may cell decompose U subjacent to these definable sets. Let V be an open cell in this cell decomposition. Then for some sequence $(d_1, \ldots, d_n)$ we have $V \subseteq \bigcap_{i=1}^{n} F_{i,d_i}$. Because V is an open cell, for each $u \in V$, we have

$$\begin{aligned} \dim_u(f^{-1}\xi_i^{-1}\{\xi(f(u))\}) &= \dim_u(V \cap f^{-1}\xi_i^{-1}\{\xi(f(u))\}) \\ &= \dim(V \cap f^{-1}\xi_i^{-1}\{\xi_i(f(u))\}) \ . \end{aligned}$$

Apply Claim 3.4.2 to conclude. ✠

Consider now the following incidence correspondence.

$$R := \{(u, g) \in U \times G : f(u) \in \pi_\Gamma(gD' \cap \mathcal{F})\}$$

Note that R is definable.

Claim 3.4.9. *We have*

$$\dim(R) = \dim(G) + 2k \ .$$

Proof of Claim: Fix the base point $* \in D' = G'/M'$ corresponding to $M' = M \cap G'$ in the coset space. For $u \in U$, let $\widetilde{u} \in \mathcal{F}$ with $\pi_\Gamma(\widetilde{u}) = u$. Let $g_0 \in G$ with $g_0 * = \widetilde{u}$. We will check that $R_u := \{g \in G : (u, g) \in R\}$ is a homogenous space for $M \times G'$ with fibers isomorphic to $G' \cap M$. Indeed, if $h \in G'$ and $m \in M$, we have $\widetilde{u} = g_0 m h h^{-1} *$, demonstrating that $f(u) \in \pi_\Gamma((g_0 m h) D' \cap \mathcal{F})$. That is, $g_0 m h \in R_u$. On the other hand, if $g \in R_u$, then we can find some h so that $g_0 * = \widetilde{u} = g h^{-1} *$. That is, $m := g_0^{-1} g h^{-1} \in M$, the stabilizer of $*$ in G. That is, $g = g_0 m h \in g_0 M G'$. We compute that for $g_1, g_2 \in G'$ and $m_1, m_2 \in M$, we have $g_0 g_1 m_1 = g_0 g_2 m_2$ only if $g_2^{-1} g_1 = m_2 m_1^{-1} =: h \in G' \cap M = M'$.

Using the fiber dimension theorem, since all fibers over U have the same dimension, $\dim(M \times G') - \dim(M \cap G')$, we now compute that

$$\begin{aligned}
\dim R &= \dim f(U) + \dim R_u \text{ for any } u \in U \\
&= \dim U + \dim(M \times G') - \dim(M \cap G') \\
&= \dim X + \dim M + (\dim G' - \dim(M \cap G')) \\
&= \dim X + \dim M + \dim S' \\
&= \dim X + (\dim G - \dim S) + \dim S' \\
&= \dim G + 2k \ .
\end{aligned}$$

✠

Abusing notation somewhat, for $g \in G$ we will also write R_g for the fiber $\{u \in U : (u, g) \in R\}$. Note that R_g is definably, complex analytically isomorphic to $f(U) \cap \pi_\Gamma(gD' \cap \mathcal{F}')$ which is a locally closed complex analytic subset of S. It follows that the o-minimal dimension of R_g is always even.

For each $i \leq \dim X$, let us define

$$B_i := \{g \in G : \dim R_g = i\} \ .$$

Claim 3.4.10. *For $i < 2k$, we have $B_i = \varnothing$.*

Proof of Claim: We have reduced through Claim 3.4.4 to the case that S is smooth. Hence, each component of $f(U) \cap \pi_\Gamma(gD' \cap \mathcal{F}')$ has complex dimension at least $\dim U + \dim D' - \dim S = k$. ✠

The set B of the statement of the theorem may be expressed as

$$B = \bigcup_{i=0}^{\dim U} B_i \ .$$

By Claim 3.4.10, we actually have

$$B = \bigcup_{i=2k}^{\dim U} B_i \ .$$

With the next claim we show that (again by shrinking U) we may arrange that $B = B_{2k}$.

Claim 3.4.11. *Possibly after shrinking U, we have $B_i = \varnothing$ for $i > 2k$.*

Proof of Claim: Suppose that $\ell > k$ and $g \in B_{2\ell}$. Then the complex analytic set $f(U) \cap \pi_\Gamma(gD' \cap \mathcal{F})$ has a component L of complex dimension ℓ. Let $(\gamma, \widetilde{\gamma}) : \Delta^\ell \to U \times gD'$ be a complex analytic map with $\mathrm{rk}(d\gamma) = \ell$ and $\pi_\Gamma \circ \widetilde{\gamma} = f \circ \gamma$. By our choice of the witnesses to the Ax-Schanuel property for S, for some $i \leq n$ and $b \in S_{2,i}$ we have

$$f \circ \gamma(\Delta^\ell) \subseteq \xi_i^{-1}\{b\} =: S_b \subsetneq S \ .$$

By our reduction from Claim 3.4.8 and the fiber dimension theorem, we have

$$\dim X = \dim U = \dim f(U) = \dim \xi_i f(U) + \dim(f(U) \cap S_b) \ .$$

Moreover, by the homogeneity of S', we also have

$$\dim S' = \dim \xi_i(S') + \dim(S' \cap S_b)$$

and, of course,

$$\dim S = \dim S_{2,i} + \dim S_b \ .$$

By our hypothesis of persistent intersection, we have

$$\dim_{\mathbb{C}} \xi_i f(U) + \dim_{\mathbb{C}} \xi_i S' = \dim S_{2,i} + k'$$

for some $k' \geq 0$.

Written in terms of o-minimal dimension this says

$$\dim \xi f(U) + \dim \xi_i S' = \dim S_{2,i} + 2k' \ .$$

Combining this equalities, we compute that

$$\dim(f(U) \cap S_b) + \dim(S' \cap S_b) = \dim S_b + 2k - 2k' \ .$$

Since, $k' \geq 0$, this means that the expected (complex) dimension of a component of $f(U) \cap S_b \cap S_b$ is at most k, but L is such a component of complex dimension greater than k. That is, L is an atypical component of the intersection inside $S_c \subsetneq S$. Applying uniform Ax-Schanuel again, we may extend the family of weakly special varieties

$$\begin{tikzcd} & S_{1,i} \arrow[dl, "\zeta_i"'] \arrow[dr, "\xi_i"] & \\ S & & S_{2,i} \end{tikzcd}$$

for $n+1 \leq i \leq n_2$ so that each such atypical component will satisfy $L \subseteq \zeta_i(\xi_i^{-1}\{b_2\}) \subsetneq S_b \subsetneq S$ for some $i \leq n_2$ and $b_2 \in S_{2,i}$.

Repeating the reductions of the earlier claims and this extension of the list of weakly special witnesses to Ax-Schanuel $\dim S + 1$ times, we reach a contradiction to the hypothesis that R_i is nonempty for some $i > 2k$. ✠

Thus, $B = B_{2k}$. So we have

$$\dim G + 2k = \dim R = \dim B_{2k} + 2k = \dim B + 2k \ .$$

Subtracting $2k$ from both sides, we conclude that $\dim B = \dim G$. Hence, by cell decomposition, B contains an open subset of G. $\square$

4. Applications

In this section we illustrate Theorem 3.4 by considering various situations in which it applies.

4.1. **Arithmetic quotients.** Our formalism is derived from that of Bakker, Klingler, and Tsimerman in [5] for the study of arithmetic quotients. They consider definable complex quotient spaces $S_{G,\Gamma,M;\mathcal{F}}$ in which the algebraic group $\mathbf{G}$ is a semisimple $\mathbb{Q}$-algebraic group, Γ is arithmetic (so commensurable with $\mathbf{G}(\mathbb{Z})$ for some / any choice of an integral model for $\mathbf{G}$), and M is compact. They often require Γ to be neat; we will return to that issue in a moment. The definable fundamental domain $\mathcal{F}$ is not chosen sufficiently carefully in [5]. A similar issue is addressed in [18] in that one needs to take $\mathcal{F}$ to be constructed from a Siegel set associated to a maximal compact subgroup of G containing M. We understand that a detailed erratum will be available soon.

If we drop the neatness requirement on Γ, then an arithmetic quotient need not be smooth, but because every arithmetic group has a neat subgroup of finite index, for any arithmetic quotient S we may find a smooth arithmetic quotient $\widetilde{S}$ and a finite surjective map of arithmetic quotients $\widetilde{S} \to S$.

The one point space is a clearly a terminal object in the category of arithmetic quotients and the pullback construction of Proposition 2.4 specializes to the category of arithmetic quotients. Since there are only countably many arithmetic quotients all told and at most countably many maps of algebraic groups between algebraic groups defined over the rational numbers, it follows that the weakly special varieties are well-parameterized within the category of arithmetic quotients.

The Ax-Schanuel theorem has not been proven explicitly in full generality for definable analytic maps from quasiprojective algebraic varieties to arithmetic quotients though it follows from known results. The main theorem of [6] gives the Ax-Schanuel theorem in the case that $f : X^{\mathrm{an}} \to S$ is a period mapping associated to a polarized variation of integral Hodge structure on the smooth algebraic variety X and S is a quotient of a Mumford-Tate domain by some arithmetic group containing the monodromy group of the variation of Hodge structure. Thus, this space S is an arithmetic quotient. Let us sketch how to deduce Ax-Schanuel for all definable analytic maps from quasiprojective algebraic varieties to arithmetic quotients from this theorem for period mappings. Passing to finite covers, using definable Chow as in Claim 3.4.1, and arguing by Noetherian induction, we may reduce to the case that $f : X^{\mathrm{an}} \to S$ is an inclusion of the smooth algebraic variety X into the smooth arithmetic quotient S. Interpreting D as a parameter space for Hodge structures of a given shape and pulling X back to a universal family of such Hodge structures, we treat X as the base of a polarized variation of integral Hodge structures so that the inclusion of X into S may be seen as a period mapping. We then invoke the main theorem of [6].

Our last observation in verifying Convention 3.1 and the hypotheses of Theorem 3.4 for arithmetic quotients is that if $D' \subseteq D$ is a homogeneous space for which $\pi_\Gamma(D') \subseteq S$ is a special variety, then for every $g \in \mathbf{G}(\mathbb{Q}) \cap G$, $\pi_\Gamma(gD')$ is also special. Thus the set of $g \in G$ for which $\pi_\Gamma(gD')$ is special is dense in G for the Euclidean topology.

Returning to the case where we know $f : X^{\mathrm{an}} \to S$ to be a period mapping and $f(X)$ to be Hodge generic in S, that is, not contained in any proper weakly special subvariety, the union of $f^{-1}S'$ ranging over all proper special subvarieties $S' \subsetneq S$ is called the Hodge locus. Because the special subvarieties of S come from $\mathbb{Q}$-semisimple algebraic subgroups of $\mathbf{G}$ and there are only finitely many such subgroups up to $G = \mathbf{G}(\mathbb{R})^+$-conjugacy, all special subvarieties of S come from finitely many families of homogeneous spaces in the sense of Theorem 3.4. That is, we can find finitely many homogeneous spaces $D_1, \ldots, D_n \subseteq D$ so that for any special subvariety $S' \subseteq S$ there is some $g \in G$ and $i \leq n$ with $S' = \pi_\Gamma(gD_i)$. Thus, if for some special subvariety $S' \subseteq S$ the intersection of X with S' is persistently likely, then the Hodge locus is Euclidean dense in X. In fact, we may take D_i so that $S' = \pi(gD_i)$ for some $g \in G$ and we see that the subset of the Hodge locus of the form $f^{-1} \bigcup_{h \in G, \pi_\Gamma(hD_i) \text{ special}} \pi_\Gamma(hD_i)$ is Euclidean dense in X. In [7] a theorem of a similar flavor is proven. They show that if the *typical* Hodge locus is nonempty, then it is analytically dense in X. Here the typical Hodge locus is the union of all components of $f^{-1}S'$ of expected dimension as S' ranges through the special subvarieties of S. The proof in [7] uses some elements in common with ours. Notably, Ax-Schanuel plays a central role in both proofs. To pass from a nonempty typical locus to one which is dense, they argue an analysis of Lie algebras to find enough special varieties. Such a technique is not available to us in general as we must postulate the existence of special varieties of a given shape. On the other hand, such an argument does not immediately lend itself to a study of intersections with a restricted class of special varieties.

Definability of the period mappings associated to admissible, graded polarized, variation of mixed Hodge structures has been established by Bakker, Brunebarbe, Klingler, and Tsimerman in [4] and then Ax-Schanuel for these maps was proven independently by Chiu [9] and Gao and Klingler [14]. Indeed, Chiu has established a stronger Ax-Schanuel theorem with derivatives for such period maps associated to variations of mixed Hodge structures [8]. These results give the necessary ingredients to extend our result on the density of Hodge loci to variations of mixed Hodge structures. We will return to the special case of universal abelian schemes over moduli spaces in Section 4.3 to draw a conclusion from the combination of our Theorem 3.4 and Ax-Schanuel in the context of mixed Shimura varieties.

4.2. **Modular varieties.** For the sake of illustration, let us consider a very special case of Theorem 3.4. We take $S = \mathbb{A}^n = X_0(1)^n$. That is, S is affine n-space (for some positive integer n) regarded as the coarse moduli space of n-tuples of elliptic curves. We may see S as an arithmetic quotient space, taking $\mathbf{G} = \mathrm{PGL}_2^n$, $\Gamma = \mathrm{PGL}_2^n(\mathbb{Z})$, and the homogeneous space D may be identified with $\mathfrak{h}^n$ where $\mathfrak{h} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ is the upper half plane.

If $\sigma = \langle \sigma_1, \ldots, \sigma_m \rangle$ is a finite sequence taking values in $\{1, \ldots, n\}$ we may define $\pi_\sigma : S \to \mathbb{A}^m$ by $(x_1, \ldots, x_n) \mapsto (x_{\sigma_1}, \ldots, x_{\sigma_n})$. For $J \subseteq \{1, \ldots, n\}$ we list the elements of J in order as $J = \{j_1 < j_2 < \ldots < j_m\}$ and write π_J for $\pi_{\langle j_1, \ldots, j_m \rangle}$. For a singleton $J = \{j\}$, we just write π_j for π_J.

For $S' \subseteq S$ a special subvariety of S provided that for each $i \leq n$ the projection map $\pi_i : S \to \mathbb{A}^1$ is dominant, then S' defines a partition $\Pi(S')$ of $\{1, \ldots, n\}$ by the rule that i and j lie in a common element of the partition if and only if $\dim \pi_{\langle i,j \rangle} S' = 1$. Given a partition Π of $\{1, \ldots, n\}$ we say that S' is a special variety of type Π if $\Pi(S') = \Pi$. Let us observe that a special variety of type Π has dimension equal to $\#\Pi$.

Fix a partition Π with $\#\Pi = m$. Let $D'_\Pi \subseteq D$ be the homogeneous subspace of D defined by $\tau_i = \tau_j$ if and only there is some $\nu \in \Pi$ with $\{i, j\} \subseteq \nu$. Then $\pi_\Gamma(D'_\Pi) =: S'$ is the corresponding multi-diagonal subvariety of S and is a special variety of type Π. Indeed, the special varieties of the form $\pi_\Gamma(gD'_\Pi)$ as g ranges through $\mathrm{PGL}_2(\mathbb{Q}) \cap G$ are exactly the special varieties of type Π.

For a partition Π of $\{1,\ldots,n\}$ and subset $J \subseteq \{1,\ldots,n\}$ of $\{1,\ldots,n\}$, we define

$$\Pi \upharpoonright J := \{\nu \cap J : \nu \in \Pi, \nu \cap J \neq \varnothing\}$$

to be the restriction of the partition Π to J.

It is easy to check that for any special variety $S' \subseteq S$ and subset $J \subseteq \{1,\ldots,n\}$, the projection $\pi_J(S')$ is a special variety and $\Pi(\pi_J(S')) = \Pi \upharpoonright J$. With these combinatorial preliminaries in place, we may state the specialization of Theorem 3.4 to the case of $Y_0(1)^n$.

Proposition 4.1. *Let $n \geq 1$ be a positive integer and Π a partition of $\{1,\ldots,n\}$. If $X \subseteq \mathbb{A}^n$ is an irreducible complex algebraic subvariety of affine n-space, regarded as the coarse moduli space of n-tuples of elliptic curves, and for every $J \subseteq \{1,\ldots,n\}$ we have $\#\Pi \upharpoonright J + \dim \pi_J(X) \geq \#J$, then*

$$X \cap \bigcup_{S' \subseteq \mathbb{A}^n \text{ special of type } \Pi} S'$$

is dense in X for the Euclidean topology.

Proof. Let us check that the intersection between X and $S' := \pi_\Gamma(D_\Pi)$ is persistently likely. Let

$$\begin{tikzcd} & S_1 \arrow[dl, "\zeta"'] \arrow[dr, "\xi"] & \\ S & & S_2 \end{tikzcd}$$

be a pair of surjective maps of arithmetic quotients with ζ finite. The arithmetic quotients S_1 and S_2 will take the form $S_1 = S_{\mathrm{PGL}_2^n,\Gamma_1,M_1;\mathcal{F}_1}$ and $S_2 = S_{\mathrm{PGL}_2^k,\Gamma_2,M_2;\mathcal{F}_2}$ with $k \leq n$ where Γ_j is an arithmetic group in for $j = 1$ and 2 and the corresponding homogeneous spaces are $\mathfrak{h}^n$ and $\mathfrak{h}^k$, respectively. Since each PGL_2 factor is simple, the maps of algebraic groups corresponding ζ and ξ are given by coordinate projections followed by an inner automorphism defined over $\mathbb{Q}$. That is, the map of groups corresponding to ξ is given by $\langle g_1,\ldots,g_n\rangle \mapsto \langle g_{j_1},\ldots,g_{j_k}\rangle$ followed by an inner automorphism of PGL_2^k defined over $\mathbb{Q}$ for some collection of k distinct numbers $j_1,\ldots,j_k$ between 1 and n, and likewise for ζ. Let $J = \{j_1,\ldots,j_k\}$, then permuting coordinates we see that this family of weakly special varieties fits into the commuting square

$$\begin{tikzcd} & S_1 \arrow[dl, "\zeta"'] \arrow[dr, "\xi"] & \\ S \arrow[dr, "\pi_J"'] & & S_2 \arrow[dl, "\widetilde{\zeta}"] \\ & \mathbb{A}^k & \end{tikzcd}$$

where $\widetilde{\zeta} : S_2 \to \mathbb{A}^k$ is finite. We then have

$$\begin{aligned} \dim \xi\zeta^{-1}X + \xi\zeta^{-1}S' &= \dim \pi_J X + \dim \pi_J(S') \\ &= \dim \pi_J X + \#\Pi \upharpoonright J \\ &\geq k \\ &= \dim S_2 \end{aligned}$$

Since $\mathrm{PGL}_2^n(\mathbb{Q})^+$ is the commensurator of $\mathrm{PGL}_2^n(\mathbb{Z})$ and is dense in $G = \mathrm{PGL}_2^n(\mathbb{R})^+$, the concluding "in particular" clause of Theorem 3.4 applies and we find that the intersections of X with special varieties of type Π is dense in X in the Euclidean topology. $\square$

Instances of Proposition 4.1 appear in the literature. Habegger shows in [15, Theorem 1.2] that if $X \subseteq \mathbb{A}^2$ is a curve defined over the algebraic numbers, then there is a constants $c = c(X) > 0$ and $p_0(X) > 0$ so that for every prime number $p > p_0(X)$ there is an algebraic point $P \in X(\mathbb{Q}^{\mathrm{alg}}) \cap Y_0(p)(\mathbb{Q}^{\mathrm{alg}})$ with logarithmic height $h(P) \geq c \log(p)$ where here $Y_0(p)$ is the modular curve parametrizing the isomorphism classes of pairs of elliptic curves $\langle E, E' \rangle$ for which there is an isogeny $E \to E'$ of degree p. Habegger's result implies in particular that for $n = 2$ and $\Pi = \{\{1, 2\}\}$, if $X \subseteq \mathbb{A}^2_{\mathbb{Q}^{\mathrm{alg}}}$ is an affine plane curve defined over the algebraic numbers, then the intersection of X with the special varieties of type Π is Zariski dense in X. Using equidistribution results, this Zariski density could be upgraded to Euclidean density.

In the discussion after Remark 3.4.5 in [27], Zannier sketches an argument showing that if $X \subseteq \mathbb{A}^2$ is a rational affine plane curve, then the intersections of X with special curves of type Π, as in the previous paragraph, are dense in X in the Euclidean topology.

4.3. **Torsion in families of abelian varieties.** If $\pi : A \to B$ is an abelian scheme of relative dimension g over the irreducible quasiprojective complex algebraic variety B and $X \subseteq A$ is a quasi-section of π, by which we mean that π restricts to a generically finite map on X, then under some mild nondegeneracy conditions, we expect that the set

$$\pi(X \cap A_{\mathrm{tor}}) = \{b \in B(\mathbb{C}) : (\exists n \in \mathbb{Z}_+) X_b \cap A_b(\mathbb{C})[n] \neq \varnothing\}$$

of points on the base over which X meets the torsion subgroup of the fiber is dense in B if and only if $\dim B \geq g$. Masser-Zannier prove in [17] that when $B = \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ and $\pi : A \to B$ is the square of the Legendre family of elliptic curves defined in affine coordinates by $y_1^2 = x_1(x_1 - 1)(x_1 - \lambda)$ and $y_2^2 = x_2(x_2 - 1)(x_2 - \lambda)$ where λ ranges over B, and X is the curve defined by $x_1 = 2$ and $x_2 = 3$, then the set $\pi(X \cap A_{\mathrm{tor}})$ is finite. This theorem sparked much work on torsion in families of abelian varieties culminating in a result announced by Gao and Habegger that, at least for such abelian schemes $\pi : A \to B$ defined over $\mathbb{Q}^{\mathrm{alg}}$, if $X \subseteq A$ is an algebraic variety, also defined over $\mathbb{Q}^{\mathrm{alg}}$ so that the group generated by X is Zariski dense in A and $\pi(X \cap A_{\mathrm{tor}})$ is Zariski dense in B, then $\dim X \geq g$.

In the opposite direction, André, Corvaja, and Zannier study in [1] the problem of density of torsion through an analysis of the rank of the Betti map. In an appendix to that paper written by Gao, it is shown that if $\pi : A \to B$ is a principally polarized abelian scheme of relative dimension g which has no non-trivial endomorphism (on any finite covering), and for which the image of S in the moduli space $\mathcal{A}_g$ of abelian varieties of dimenion g itself has dimension at least g and $X \subseteq A$ is the image of a section of π, then $\pi(X \cap A_{\mathrm{tor}})$ is dense in B in the Euclidean topology. The proof of this result made use of the Ax-Schanuel theorem for pure Shimura varieties and was subsequently upgraded. See in particular Gao's work on the Ax-Schanuel theorem for the universal abelian variety [12] and on the Betti map in [11, 13].

Gao's main theorem, Theorem 1.1, in [11] may be seen as a geometric elaboration of what Theorem 3.4 means for the density of torsion. Gao considers an abelian scheme $\pi : A \to B$ of relative dimension g over a quasiprojective complex algebraic variety B and a closed irreducible subvarierty $X \subseteq A$ and then establishes the conditions under which the generic rank of the Betti map restricted to X may be smaller than expected. It is noted with [1, Proposition 2.2.1] that density of the torsion in X follows from the Betti map, generically, having rank $2g$ on X. Thus, the converse of Gao's condition gives a criterion for when the torsion is dense.

In more detail, taking finite covers if necessary, one may pass from the problem of density of torsion in X as a subvariety of A, to the density of torsion in $\widetilde{\iota}(X)$ in $\mathfrak{A}$ where $\mathfrak{A} \to \mathcal{A}$ is a universal abelian variety over a moduli space $\mathcal{A}$ of abelian varieties of some fixed polarization type with some fixed level structure and the Cartesian square

$$\begin{tikzcd} A \arrow[r, "\widetilde{\iota}"] \arrow[d, "\pi"'] & \mathfrak{A} \arrow[d, "\pi"] \\ B \arrow[r, "\iota"] & \mathcal{A} \end{tikzcd}$$

expresses $A \to B$ as coming from this universal family. To ease notation, we replace B by $\iota(B)$ and X by $\widetilde{X}$. Shrinking the moduli space, possibly taking covers, and moving to an abelian subscheme of $\mathfrak{A}_B$, we may arrange that X is not contained in any proper weakly special varieties. At this point, Theorem 3.4 says that $X \cap \mathfrak{A}_{\mathrm{tor}}$ is dense in X in the Euclidean topology if the intersection of X with the zero section is persistently likely. Gao's criterion expresses geometrically what persistent likelihood means here: for any abelian subscheme $\mathfrak{A}'$ of $\mathfrak{A}_B$, if $p : \mathfrak{A}_B \to \mathfrak{A}_B/\mathfrak{A}'$ is the quotient map, then $\dim p(X)$ is at least the relative dimension of $\mathfrak{A}_B/\mathfrak{A}'$ over B.

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