Dynamical Mordell-Lang problems

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Discrete algebraic dynamics

Definition

A discrete algebraic dynamical system over a field K is a pair (X, f) consisting of an algebraic variety X over K and a morphism of algebraic varieties $f : X \to X$.

We may wish to relax this definition is several directions.

- We might allow f to be merely a rational function or even just an algebraic correspondence.
- We might want to consider a dynamical system given by a semigroup of operators.
- We might want to take X and f simply to be definable relative to some other theory.

Basic dynamical Mordell-Lang problem

Problem

Given an algebraic dynamical systems $f : X \to X$ over a field K, a point $a \in X(K)$ and a subvariety $Y \subseteq X$, describe $\{n \in \mathbb{N} : f^{\circ n}(a) \in Y(K)\}$

Why is this called a Mordell-Lang problem?

Let

- X be an abelian variety,
- $f : X \to X$ be given by translation by some element $\gamma \in X(K)$, and
- a be the identity element of X,

then $\mathcal{O}_f(a) := \{f^{\circ n}(a) : n \in \mathbb{N}\}$ is the monoid generated by γ.

The usual Mordell-Lang conjecture (when the characteristic of K is zero) asserts in this case that $Y(K) \cap \mathcal{O}_f(a)$ is a finite union of translates of submonoids of $\mathcal{O}_f(a)$. In particular, $\{n \in \mathbb{N} : f^{\circ n}(a) \in Y(K)\}$ is a finite union of points and arithmetic progressions.

This rank one problem is a very special case of the Mordell-Lang conjecture and was already addressed in the 1930s and 1940s by Chaubuty, Skolem, Lech and Mahler.

Skolem's method

Theorem

Let X be a commutative algebraic group over a field K of characteristic zero, $\gamma \in X(K)$, and $Y \subseteq X$ a closed subvariety. Then the set $\{n \in \mathbb{Z} : [n]_X(\gamma) \in Y(K)\}$ is a finite union of points and arithmetic progressions.

Proof of theorem

Proof.

- Choosing equations for X, γ, and Y over some finitely generated subring of K, we may assume that K is a p-adic field and that all of the named objects have good integral models.
- Using the theory of p-adic Lie groups, we can find a p-adic analytic exponential map and thereby produce a p-adic analytic function $E : \mathbb{Z}_p \to X(\mathbb{Q}_p)$ satisfying $E(n) = [n]_X(\gamma)$ for $n \in \mathbb{N}$.
- The set $\{x \in \mathbb{Z}_p : E(x) \in Y(\mathbb{Q}_p)\}$ is then the zero set of a p-adic analytic function on $\mathbb{Z}_p$ and is therefore a finite union of points and cosets of $p^n\mathbb{Z}_p$.

$\square$

Skolem's method for general dynamical systems

Theorem (Ghioca, Tucker)

If $\Phi : X \to X$ is a self-map of a smooth algebraic variety over a p-adic field K, $Y \subseteq X$ is a closed subvariety, $P \in X(K)$, $\lim_{n \to \infty} \Phi^{\circ n}(P) =: Q$, and $d\Phi_Q : T_Q X \to T_Q X$ is diagonalizable with all of its eigenvalues having p-adic absolute value less than one, then $\{n \in \mathbb{N} : \Phi^{\circ n}(P) \in Y(K)\}$ is a finite union of points and arithmetic progressions.

Skolem's method for general dynamical systems

Problem

Given an algebraic dynamical system $f : X \to X$ over some finitely generated field K of characteristic zero and point $a \in X(K)$ is it always possible to find some $\mathfrak{p}$-adic completion of K and a p-adic analytic function $E : \mathbb{Z}_p \to X(K_\mathfrak{p})$ so that $E(n) = f^{\circ n}(a)$ for all $n \in \mathbb{N}$?

- Even when $X = \mathbb{P}^1$ and f is an ordinary rational function of one variable, this is still an open problem.
- The issue is that one wishes to uniformize f near a fixed point, but it is not obvious that there are any appropriate fixed or periodic points in the closure of the orbit of a.

Cautionary remarks on p-adic method

Benedetto, Ghioca, and Tucker have given examples of

- polynomials $f_1, \dots, f_n$,
- p-adic number $a_1, \dots, a_n$, and
- a p-adic analytic function $G(x_1, \dots, x_n)$

for which $\lim f_i^{\circ m}(a_i)$ exists for each $i \leq m$, but $\{m \in \mathbb{N} : G(f_1^{\circ m}(a_1), \dots, f_n^{\circ m}(a_n)) = 0\}$ is infinite but does not contain an arithmetic progression.

On the other hand, they do prove that the sequence of such zeros grows very quickly.

A real analytic version of Skolem's method?

It would seem that Skolem's method does not apply in the Euclidean topology for even if we had an analytic function E satisfying $E(n) = f^{\circ n}(a)$ for all $n \in \mathbb{N}$, since the integers are discrete in $\mathbb{R}$, from an analytic equation $F(E(n)) = 0$ for infinitely many $n \in \mathbb{N}$, we could not reach a useful conclusion.

Analytic uniformization

There are classical results from the early Twentieth Century due to König and Böttcher that univariate complex analytic dynamical systems may be expressed in standard monomial form near attracting (or repelling!) fixed points.

Theorem

If $f(x)$ is a complex analytic function near the origin with $f(0) = 0$ and $|f'(0)| \neq 1$, then there is another complex analytic function u also defined near the origin so that one of the following diagrams commutes where $\lambda := f'(0)$, $N := \mathrm{ord}_0(f)$ and Δ is a neighborhood of the origin.

$$\begin{tikzcd} \Delta \arrow[r, "f"] \arrow[d, "u"'] & \Delta \arrow[d, "u"] \\ \Delta \arrow[r, "\lambda\cdot"] & \Delta \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \Delta \arrow[r, "f"] \arrow[d, "u"'] & \Delta \arrow[d, "u"] \\ \Delta \arrow[r, "x \mapsto x^N"] & \Delta \end{tikzcd}$$

Real analytic orbits

Proposition

If f is a real analytic function near the origin, a is close to the origin, and $\lim_{n\to\infty} f^{\circ n}(a) = 0$, then there is a function F definable in $\mathbb{R}_{an,\exp}$ for which $F(n) = f^{\circ n}(a)$ for $n \in \mathbb{N}$.

Theorem

Let $f_1, \ldots, f_n$ be a finite sequence of real analytic functions each defined on some interval. Let $a_1, \ldots, a_n$ be real numbers for which $\lim_{m\to\infty} f_i^{\circ m}(a_i)$ exists for each i. Then if $X \subseteq \mathbb{R}^n$ is a subanalytic set, the set $\{m \in \mathbb{Z}_+ : (f_1^{\circ m}(a_1), \ldots, f_n^{\circ m}(a_n)) \in X\}$ is a finite union of points and arithmetic progressions.

Cautionary remark about complex dynamics

Extending Skolem's method to complex analytic dynamics leads very quickly to notorious issues in diophantine approximation. It seems plausible that the the p-adic counterexamples may be adapted to the complex analytic setting.

Higher rank dynamical Mordell-Lang problem

The difficult cases of the usual Mordell-Lang conjecture concern finitely generated groups of rank greater than one. While the dynamical Mordell-Lang problem for a single map is already nontrivial, the higher rank case is much more complicated. From point of view of understanding the induced structure on dynamical orbits, the simplest higher rank problem is to understand algebraic relations between different points of single orbit.

Problem

Given an algebraic dynamical systems $f_i : X_i \to X_i$ (for $i \leq n$) over a field K and a points $a_i \in X_i(K)$ describe for $Y \subseteq \prod_{i=1}^n X_i$ an algebraic subvariety, describe
$\{(m_1, \ldots, m_n) \in \mathbb{N}^n : (f_1^{\circ m_1}(a_1), \ldots, f_n^{\circ m_n}(a_n)) \in Y(K)\}$

Specializing further, we might want to take $X_i = X$, $f_i = f$, and $a_i = a$ to be the same for all $i \leq n$.

Some observations

- In some cases, the induced structure is very chaotic. For instance, if $X_i = \mathbb{A}^1_{\mathbb{C}}$, $f_i(x) = x + 1$, and $a_i = 0$, then we are studying integer points on varieties.
- The generic case may be analyzed by considering instead the difference equations $\sigma_i(x_i) = f_i(x_i)$.

Nonarchimedian local theorem via Mann property

We specialize the higher dimensional problem to the case of nonarchimedian analysis.
Fix

- (K, v) a complete nonarchimedian field with ring of integers R,
- an analytic function f over R satisfying $f(0) = 0$ and $v(f'(0)) > 0$, and
- a nonzero point $a \in R$ with $\lim f^{\circ m}(a) = 0$.

What are the possible analytic relations on $\mathscr{O}_f(a)$?

Warm up case: $\lambda := f'(0) \neq 0$

- By the nonarchimedean version of König's uniformization, by an analytic change of variables we may assume $f(x) = \lambda x$ and $a = 1$ so that $f^{\circ m}(a) = \lambda^m$ for $m \in \mathbb{N}$.
- If $G(x_1, \ldots, x_n) = \sum g_\alpha x^\alpha \in R[[x_1, \ldots, x_n]]$ is any nonzero power series, then there is a finite set S of multi-indices for which for any $b \in R^n$, $\min\{v(g_\beta b^\beta) : \beta \in \mathbb{N}^n\} = \min\{v(g_\alpha b^\alpha) : \alpha \in S\}$.
- Consequently, if $G(f^{\circ m_1}(a), \ldots, f^{\circ m_n}(a)) = 0$, then for some $\beta \neq \alpha \in S$, $\sum_i (\alpha_i - \beta_i) v(\lambda) m_i = v(g_\beta / g_\alpha)$ must hold.
- If such a valuation equation is consistent, then there is some $\mu \in \langle \lambda \rangle$, so that for $\vec{m}$ satisfying the linear equation, $\lambda^{\vec{m}}$ satisfies $X^\alpha = \mu X^\beta$.
- Thus, the analytic varieties meeting $\mathscr{O}_f(a)^n$ densely are (analytic reparametrizations of) finite unions of translates of algebraic tori.

Superattracting case: $f'(0) = 0$

- This time, at least in characteristic zero and perhaps at the cost of replacing a with $f^{\circ m}(a)$ for some m, we may use the analogue of Böttcher's theorem to conjugate f to x^N so some $N > 1$ so that $f^{\circ m}(a) = a^{N^m}$ for $m \in \mathbb{N}$ [though, to be honest, this reduction is unnecessary].
- As before, analytic relations on the orbit give rise to linear relations of the form

$$\sum v(f^{\circ m_i}(a))(\alpha_i - \beta_i) = v(d)$$

- That is, inhomogeneous equations of the form $\sum r_i N^{m_i} = s$.
- Not only does the Mordell-Lang property hold for the multiplicative group generated by N, but it has the Mann property (isolated by van den Dries and Günaydın).

Super-attracting case continued with the Mann property

Definition

Let L be a field. The group $\Gamma \leq L^\times$ has the Mann property if for any linear function $L(x_1, \ldots, x_n) = \sum_{i=1}^n a_i x_i \in L[x_1, \ldots, x_n]$ there are only finitely many non-degenerate solutions to $L(\gamma_1, \ldots, \gamma_n) = 1$ with $\gamma_i \in \Gamma$ where a solution is degenerate if $\sum_{i \in I} a_i \gamma_i = 0$ for some $I \subsetneq \{1, \ldots, n\}$

It follows that every system of (inhomogeneous) linear equations in Γ is reducible to a system defined by equations of the form $x_i = \gamma x_j$ or $x_k = \delta$ for γ and δ from Γ

Returning to the dynamical problem, these valuations equations correspond to the analytic equations $f^{\circ \ell}(x_j) = x_i$ or $x_k = f^{\circ \ell}(a)$.

It follows that all analytic relations on $\mathscr{O}_f(a)$ are finite unions of relations defined by conjunctions of such equations.

Concluding speculations

- Is the trichotomy theorem for difference equations reflected in a trichotomy for the induced structure on dynamical orbits?
- To what extent do these analytic arguments extend to higher dimensional dynamical systems?
- Does the o-minimal analysis give useful information about relations on dynamical orbits beyond dimension one?