DIFFERENTIALLY VALUED FIELDS ARE NOT DIFFERENTIALLY CLOSED

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Abstract. In answer to a question of L. van den Dries, we show that no differentially closed field possesses a differential valuation.

1. Introduction

In connection with his work on H-fields [1], L. van den Dries asked whether a differentially closed field can admit a nontrivial (Rosenlicht) differential valuation.

If K is a field and v is a Krull valuation on K and L/K is an extensions field, then there is at least one extension of v to a valuation on L. It is known that the analogous statement for differential specializations on differential fields is false. Indeed, anomalous properties of specializations of differential rings were observed already by Ritt [10] and examples of nonextendible specializations are known (see Exercise 6(c) of Section 6 of Chapter IV of [6] and [3, 4, 8] for a fuller account).

In this short note, we answer van den Dries' question negatively by exhibiting a class of equations which cannot be solved in any differentially valued field even though they have solutions in differentially closed fields.

I thank M. Aschenbrenner and L. van den Dries for bringing this question to my attention and discussing the matter with me and the Isaac Newton Institute for providing a mathematically rich setting for those discussions.

2. Logarithmic derivatives and differential valuations

In this section we recall Rosenlicht's notion of a differential valuation and show how the elliptic logarithmic derivative construction can be used to answer van den Dries' question.

In what follows, if v is a valuation on a field K, then we write $\mathcal{O} := \{x \in K \mid v(x) \geq 0\}$ for the v-integers and if ∂ is a derivation on K, then we write $\mathcal{C} := \{x \in K \mid \partial(x) = 0\}$ for the differential constants.

Definition 2.1. A *differential valuation (in the sense of Rosenlicht)* v on a differential field (K, ∂) is a valuation for which the differential constants form a field of representatives in the sense that $\mathcal{C}^\times \subseteq \mathcal{O}^\times$ and for any $x \in K$ there is some $y \in \mathcal{C}$ with $v(x - y) > 0$ and an abstract version of L'Hôpital's Rule holds in the sense that if $v(x) > 0$ and $v(y) > 0$, then $v(y'x/x') > 0$.

Remark 2.2. It should be noted that Rosenlicht's notion of a differential valuation does not agree with Blum's [3].

Partially supported by NSF CAREER grant DMS-0450010.

Rosenlicht proved that to a differential valuation there is an associated *asymptotic couple*: the value group of the valuation, Γ, given together with a function $\psi : \Gamma \smallsetminus \{0\} \to \Gamma$ defined by $\psi(v(a)) = v(\partial(a)/a)$ for $a \in K^\times$ with $v(a) \neq 0$ [11]. For us, the most important property of this asymptotic couple is that if $\alpha, \beta \in \Gamma \smallsetminus \{0\}$, then $\psi(\alpha) < \psi(\beta) + |\beta|$. From this property it follows that if a differential field admits a differential valuation, then the derivation may be scaled so that the resulting derivation preserves the ring of integers.

Lemma 2.3. *If (K, ∂) is a differential field and v is a nontrivial differential valuation on K, then there is some $b \in K^\times$ so that is $\widetilde{\partial} = b\partial$, then v is a differential valuation on $(K, \widetilde{\partial})$ for which $\widetilde{\partial}(\mathcal{O}) \subseteq \mathcal{O}$.*

Proof. Let $a \in K^\times$ be any element with $v(a) \neq 0$ and set $b := a/\partial(a)$. If $x \in \mathcal{O}$, then we may write $x = c + y$ where $\partial(c) = 0$ and $v(y) > 0$. Using Rosenlicht's inequality, we have $\psi(v(a)) = -v(b) < \psi(v(y)) + v(y) = v(\partial(y)) = v(\partial(c + y)) = -v(b) + v(\widetilde{\partial}(x))$. In particular, $v(\widetilde{\partial}(x)) > 0$. $\square$

With the next lemma we note that scaling a derivation does not change the property of the differential field being differentially closed.

Lemma 2.4. *If (K, ∂) is a differentially closed field, $b \in K^\times$ is nonzero, and $\widetilde{\partial}$, then $(K, \widetilde{\partial})$ is also differentially closed.*

Proof. Employing the Blum axioms for differentially closed fields [2], we must show that if $P(X_0, \ldots, X_n) \in K[X_0, \ldots, X_n]$ is an irreducible polynomial over K in $n + 1$ variables and $G(X_0, \ldots, X_{n-1}) \in K[X_0, \ldots, X_{n-1}]$ is a polynomial in fewer variables, then there is some $a \in K$ with $P(a, \widetilde{\partial}(a), \ldots, \widetilde{\partial}^n(a)) = 0$ and $G(a, \ldots, \widetilde{\partial}^{n-1}(a)) \neq 0$.

In the ring $K\langle\partial\rangle$ of linear differential operators in ∂ over K, for each positive integer m we may write $(b\partial)^m = b^m \partial^m + \sum_{i=1}^{m-1} d_i^{(m)} \partial^i$ for some $d_i^{(m)} \in K$. Indeed, the base case of $m = 1$ is trivial, and $(b\partial)^{m+1} = b\partial(b^m\partial^m + \sum_{i=1}^{m-1} d_i^{(m)}\partial^i) = b(b^m\partial^{m+1} + mb^{m-1}\partial(b)\partial^m + \sum_{i=1}^{m-1}(\partial(d_i^{(m)})\partial^i + d_i^{(m)}\partial^{i+1})) = b^{m+1}\partial^{m+1} + \sum_{i=1}^{m} d_i^{(m+1)}\partial^j$ where $d_m^{(m+1)} = mb^m\partial(b) + bd_{m-1}^{(m)}$ and $d_i^{(m+1)} = \partial(d_i^{(m)}) + d_{j-1}^{(m)}$ for $1 \leq j < m$.

The map $\rho : K[X_0, \ldots, X_n] \to K[X_0, \ldots, X_n]$ given by $X_0 \mapsto X_0$ and $X_i \mapsto b^i X_i + \sum_{j=1}^{i-1} d_j^{(i)} X_i$ is an automorphism for which for any $F \in K[X_0, \ldots, X_n]$ and $c \in K$ we have $\rho(F)(c, \partial(c), \ldots, \partial^n(c)) = F(c, \widetilde{\partial}(c), \ldots, \widetilde{\partial}^n(c))$. As P is irreducible and ρ is an automorphism, $\rho(P)$ is irreducible. Visibly, $\rho(K[X_0, \ldots, X_{n-1}]) \subseteq K[X_0, \ldots, X_{n-1}]$. So, $\rho(G) \in K[X_0, \ldots, X_{n-1}]$. As (K, ∂) is differentially closed there is some $d \in K$ with $0 = \rho(P)(d, \partial(d), \ldots, \partial^n(d)) = P(d, \widetilde{\partial}(d), \ldots, \widetilde{\partial}^n(d))$ and $0 \neq \rho(G)(d, \ldots, \partial^{n-1}(d)) = G(d, \ldots, \widetilde{\partial}^{n-1}(d))$. That is, $(K, \widetilde{\partial})$ is differentially closed. $\square$

Theorem 2.5. *Suppose that (K, ∂) is a differentially closed field and v is a valuation on K for which the derivation preserves the ring of integers in the sense that $\partial(\mathcal{O}) \subseteq \mathcal{O}$. Then v is trivial.*

Proof. Let E be any elliptic curve over $\mathcal{C} \cap \mathcal{O}$. If v is trivial on $\mathbb{Q}$, we can take E to be any elliptic curve over $\mathbb{Q}$. Otherwise, v restricts to a p-adic valuation on $\mathbb{Q}$ and we can take E to be a model of an elliptic curve over $\mathbb{Z}$ having good reduction at p.

Consider the elliptic logarithmic derivative $\partial \log_E : E(K) \to \mathbb{G}_a(K)$. The reader should consult section 22 of chapter 5 of [6] for a thorough development of the theory of logarithmic differentiaion. For the sake of completeness we recall the construction of $\partial \log_E$.

There is a group homomorphism $\nabla : E(K) \to TE(K)$ from the K-rational points of E to the K-rational points of the tangent bundle of E defined in coordinates by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n; \partial(x_1), \ldots, \partial(x_n))$. As we will need to keep track of integrality conditions, it will help to see ∇ more conceptually. Using the Weil restriction of scalars construction, one can identify $TE(K)$ with $E(K[\epsilon]/(\epsilon^2))$, or more generally, $TE(R)$ with $E(R[\epsilon]/(\epsilon^2))$ for any commutative $\mathcal{C} \cap \mathcal{O}$-algebra R. If we have a derivation $\delta : R \to R$ on the algebra R, then there is a ring homomorphism $\exp(\delta) : R \to R[\epsilon]/(\epsilon^2)$ given by $x \mapsto x + \delta(x)\epsilon$. The map $\nabla : E(R) \to TE(R)$ corresponds to the map on points $E(R) \to E(R[\epsilon]/(\epsilon^2))$ induced by $\exp(\delta)$. In particular, ∇ takes R-rational points to R-rational points.

The tangent bundle TE splits as $s : TE \to E \times T_0E$ where T_0E is the tangent space to E at the origin via the map $(P, w) \mapsto (P, d\tau_{-P}w)$ where $\tau_{-P} : E \to E$ is the translation map $x \mapsto x - P$ on E. If $\pi : E \times T_0E \to \mathbb{G}_a$ is the projection onto the second coördinate followed by an isomorphism between T_0E and the additive group $\mathbb{G}_a$, then the elliptic logarithmic derivative is $\partial \log_E = \pi \circ s \circ \nabla$.

As K is differentially closed, the map $\partial \log_E : E(K) \to \mathbb{G}_a(K) = K$ is surjective. Indeed, one can see this in several ways. For instance, we can work with the Lascar rank (see [7]). The kernel of $\partial \log_E$ is $E(\mathcal{C})$ and as such has Lascar rank 1 whilst the Lascar rank of $E(K)$ is ω. Hence, the Lascar rank of the image of $\partial \log_E$ is also ω which is the same as that of the connected group $\mathbb{G}_a(K)$. Hence, $\partial \log_E$ is surjective. Alternatively, one could simply apply the geometric axioms of [9]. Let $P \in \mathbb{G}_a(K)$ be any point. Relative to the above trivialization of TE, we define a section $s_P : E \to TE$ by $Q \mapsto (Q, P)$. By the geometric axiom, there is a point $Q \in E(K)$ with $s_P(Q) = s \circ \nabla(Q)$. That is, $P = \partial \log_E(Q)$.

Since each of the maps forming $\partial \log_E$ takes $\mathcal{O}$-rational points to $\mathcal{O}$-rational points, the image of $\partial \log_E$ on $E(\mathcal{O})$ is contained in $\mathbb{G}_a(\mathcal{O}) = \mathcal{O}$. As E is proper, $E(\mathcal{O}) = E(K)$ (or, really, the image of $E(\mathcal{O})$ in $E(K)$ under the map induced by $\mathcal{O} \hookrightarrow K$ is all of $E(K)$). Hence, $\mathcal{O} = K$. That is, v is a trivial valuation. □

Combining Lemmata 2.3 and 2.4 with Theorem 2.5 we conclude with a negative answer to van den Dries question.

Corollary 2.6. *No differentially closed field admits a nontrivial differential valuation.*

Remark 2.7. As with Buium's construction of examples of Ritt's anomaly of the differential dimension of an intersection [5], our construction is based on the observation that projective algebraic varieties may admit nonconstant differential regular functions. Indeed, as the reader can readily verify, our argument shows that if (K, ∂) is a differential field admitting a nontrivial valuation whose ring of integers is preserved by ∂, and X is a projective scheme over $\mathcal{O}$ whose Albanese map is injective, then by composing the Albanese map with a component of a Manin homomorphism one produces a nonconstant differential regular function $f : X(K) \to \mathbb{A}^1(K)$ whose image is contained in $\mathcal{O}$.

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