

EFFECTIVE DIFFERENCE ELIMINATION AND NULLSTELLENSATZ

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ABSTRACT. We prove effective Nullstellensatz and elimination theorems for difference equations in sequence rings. More precisely, we compute an explicit function of geometric quantities associated to a system of difference equations (and these geometric quantities may themselves be bounded by a function of the number of variables, the order of the equations, and the degrees of the equations) so that for any system of difference equations in variables $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_m)$, if these equations have any nontrivial consequences in the \mathbf{x} variables, then such a consequence may be seen algebraically considering transforms up to the order of our bound. Specializing to the case of $n = 0$, we obtain an effective method to test whether a given system of difference equations is consistent.

1. INTRODUCTION

We say that a sequence of complex numbers $(a_j)_{j=0}^\infty$ satisfies a difference equation with constant coefficients if there is a nonzero polynomial $F(x_0, \dots, x_e) \in \mathbb{C}[x_0, \dots, x_e]$ such that, for every natural number j , the equation $F(a_j, a_{j+1}, \dots, a_{j+e}) = 0$ holds. This can also be defined for systems of difference equations in several variables. Such difference equations and the sequences that solve them are ubiquitous throughout mathematics and in its applications to the sciences, including such areas as combinatorics, number theory, and epidemiology, amongst many others (see Section 4 for some of the examples).

In this paper we resolve some fundamental problems about difference equations. The questions we answer include the following (for precise statements, including the way non-constant coefficients can appear, see Section 3):

- (1) Under what conditions does a system of difference equations have a sequence solution?
- (2) Can these conditions be made sufficiently transparent to allow for efficient computation?
- (3) Given a system of difference equations on $(n + m)$ -tuples of sequences, how does one eliminate some of the variables so as to deduce the consequences of these equations on the first n variables?

Our solution to the first question is a conceptual difference Nullstellensatz, to the second, an effective difference Nullstellensatz, and to the third, an effective difference elimination algorithm. Even though the abstract Nullstellensatz is intellectually satisfying in that conditions of different kinds are shown to be equivalent, namely the existential condition that there is a sequence solution to a system of difference equations and the universal condition that the difference ideal generated by the equations is proper, the difficult work and applications, both theoretical and practical, comes with our main effective theorems.

Effective elimination theorems and methods have a long history and play central roles in computational algebra. Row reduction, or Gaussian elimination, is a fundamental technique in linear algebra. Elimination for polynomial equations is substantially more complicated and has been the subject of intensive and sophisticated work [5, 23, 22]. In recent work of the first two authors joined by Vo [28], effective elimination theorems were obtained for algebraic differential equations through a reduction to the polynomial case through the decomposition-elimination-prolongation method.

While these questions are important and difference equations have been studied intensively both for their applications and theory, to our knowledge, none of these questions has received a satisfactory answer in the literature. We explain below how some known results, both positive and negative, may help explain the existence of this lacuna. In particular, in some essential ways, the effective Nullstellensatz and elimination problems for difference equations are substantially more difficult than the corresponding problems for differential equations and the methods of [28] do not routinely transpose to this context.

The foundational work on difference algebra, that is, the study of the theory of difference rings and of difference equations as encoded through the algebraic properties of rings of difference polynomials, was initiated by Cohn in [7], following the tradition of Ritt and Kolchin in differential algebra. Deep results have been obtained in this subject, but their relevance to the problems at hand is hampered by their restrictions, for the Nullstellensatz and elimination theorems, to the case in which solutions are sought in difference *fields*, and thus have little bearing on the structures used in practice, namely difference rings presented as rings of sequences, such as $\mathbb{C}^{\mathbb{N}}$ given with the shift operator $\sigma : (a_i)_{i=0}^{\infty} \mapsto (a_{i+1})_{i=0}^{\infty}$. Moreover, even if restricted to difference fields, the known elimination theorems are at best theoretically effective.

Chatzidakis and Hrushovski studied difference fields from the perspective of mathematical logic in [6]. There, they established a recursive axiomatization for the theory of existentially closed difference fields and proved a quantifier simplification theorem. From this it follows that in principle there are effective procedures to check the consistency of difference equations in difference fields and to perform difference elimination in difference fields. More recent work of Tomašić [33, 34] geometrizes the quantifier simplification theorem and brings the complexity of these algorithms to primitive recursive, though this effectivity is still theoretical — to call the implicit bounds astronomical would be a gross understatement — and a practical implementation of this work is infeasible. In symbolic computation, steps have been taken towards extending the characteristic set method from differential algebra to the study of difference and difference-differential equations in works of Gao, van der Hoeven, Li, Yuan, Zhang [14, 15, 27, 26]. These methods are more efficient than those coming from logic, but as they are restricted to the study of inversive prime difference ideals, they, too, are fundamentally results about solutions to difference equations in difference fields and the constructions of difference resultants depend on restrictive hypotheses.

The situation for difference equations in sequence rings differs starkly. Simple examples show that consistency checking in difference fields is not the same problem as consistency checking for sequences. For example, the system of difference equations $x\sigma(x) = 0$, $x + \sigma(x) = 1$ has no solution in a difference field, but the sequence $0, 1, 0, 1, \dots$ is a solution in $\mathbb{C}^{\mathbb{N}}$.

More seriously, theorems of Hrushovski and Point [21] show that the logical methods used for difference fields fail dramatically for sequence rings. In particular, they show that the first-order theory of $\mathbb{C}^{\mathbb{N}}$ regarded as a difference ring is undecidable. Thus, we cannot derive a consistency checking method from a recursive axiomatization of this theory nor can we produce an elimination algorithm from an effective quantifier elimination theorem; no such axiomatization or quantifier elimination procedure exists. That we succeed in solving the effective consistency checking and effective elimination problems for difference equations in sequence rings is all the more surprising given these undecidability results.

Let us explain more precisely what we actually prove and where the new ideas appear in our arguments. We have two main theorems: Theorem 1 an effective Nullstellensatz and Theorem 2 an effective difference elimination theorem. Strictly speaking, the effective Nullstellensatz is

a special case of an effective elimination theorem, but we prove elimination by bootstrapping through the Nullstellensatz.

The key to our proofs is a new form of the decomposition-elimination-prolongation (DEP) method. As is completely standard, a system of difference equations may be regarded as a system of algebraic equations in more variables together with specifications that certain coordinates should be obtained from others by the application of the distinguished endomorphism and the usual DEP methods allow for one to cleverly reduce questions about the original system of difference equations to questions entirely about algebraic equations. A version of the DEP method for difference equations in difference fields is employed in [19] for the purpose of computing explicit bounds in Diophantine geometric problems. This DEP method cannot work for the problems at hand (see Section 5) but we overcome this obstacle by developing a strikingly more robust DEP method. With our theorems we explicitly bound the number of prolongations required to solve the problems at hand, *i.e.* testing a system of difference equations for consistency or computing a nontrivial element of the elimination ideal. Our bounds are quite small. Indeed, roughly speaking, for balanced systems in which the number of equations is equal to the number of variables, the bounds are very small and even for under-determined systems the bounds are humanly comprehensible, and small enough in many cases to permit efficient computation.

We draw an interesting theoretical conclusion from our work towards the explicit bounds for the difference elimination problem in Section 7. Specifically, with Proposition 4 we show that for (K, σ) any algebraically closed difference field, whenever a finite system of difference equations over K is consistent in the sense that it has a solution in some difference ring, then it already has a solution in the ring of sequences of elements of K . We give a soft proof of such a difference Nullstellensatz under the hypothesis that K is uncountable with Proposition 1. In extending this result to general K we use the geometric analysis of our proof of our main theorem and a remarkable theorem of Hrushovski on the first-order theory of the Frobenius automorphism.

The paper is organized as follows. We give the basic definitions in Section 2, and then introduce the notation and terminology specific to our paper. The main results, Theorem 1 for the effective Nullstellensatz and Theorem 2 for the effective elimination, are expressed in Section 3. In Section 4, we illustrate our results in several practical examples. With Section 5, we present counterexamples to an effective strong difference Nullstellensatz and to the application of the usual DEP method to these problems. The proofs of the main theorems are presented in Section 6. Finally, in Section 7, we strengthen the difference Nullstellensatz giving equivalent criteria for the existence of sequence solutions to systems difference equations over any algebraically closed field.

2. PRELIMINARIES

A detailed introduction to difference rings can be found in [7, 25].

Definition 1. A *difference ring* is a pair (A, σ) where A is a commutative ring and $\sigma : A \rightarrow A$ is a ring endomorphism.

Example 1. If R is any commutative ring, then the sequence rings $R^{\mathbb{N}}$ and $R^{\mathbb{Z}}$ are difference rings with σ defined by $\sigma((x_i)_{i \in \mathbb{N}}) := (x_{i+1})_{i \in \mathbb{N}}$ ($\sigma((x_i)_{i \in \mathbb{Z}}) := (x_{i+1})_{i \in \mathbb{Z}}$, respectively).

Definition 2. A *map of difference rings* $\psi : (A, \sigma) \rightarrow (B, \tau)$ is given by a map of rings $\psi : A \rightarrow B$ such that that $\tau \circ \psi = \psi \circ \sigma$.

Remark 1. We often abuse notation saying that A is a difference ring when we mean the pair (A, σ) .

Definition 3. If A is a difference ring, then the free difference A -algebra in one generator x over A , $A\{x\}$, also called the *ring of difference polynomials* in x over A , may be realized as the ordinary polynomial ring $A[\{\sigma^j(x) : j \in \mathbb{N}\}]$ in the indeterminates $\{\sigma^j(x) : j \in \mathbb{N}\}$. Iterating this procedure, one obtains the difference polynomial ring $A\{x_1, \dots, x_n\}$ in n variables.

Definition 4. For $P \in A\{x_1, \dots, x_n\}$ and $1 \leq i \leq n$, we define the *order* of P with respect to x_i , denoted $\text{ord}_{x_i}(P)$ to be the maximal h for which $\sigma^h(x_i)$ appears in P . If no $\sigma^h(x_i)$ appears, we set $\text{ord}_{x_i}(P) := -1$.

Example 2. $\text{ord}_{x_3}(\sigma^3(x_1) + x_2 + \sigma(x_3)^2 + 1) = 1$.

Definition 5. If (A, σ) is a difference ring and $S \subseteq A\{x_1, \dots, x_n\}$ is a set of difference polynomials over A , $(A, \sigma) \subseteq (B, \sigma)$ is an extension of difference rings, and $\mathbf{b} = (b_1, \dots, b_n) \in B^n$ is an n -tuple from B , then we say that \mathbf{b} is a *solution* of the equations $\{f = 0 \mid f \in S\}$ if, under the unique map of difference rings $A\{x_1, \dots, x_n\} \rightarrow B$ given by extending the given map $A \rightarrow B$ and sending $x_i \mapsto b_i$ for $1 \leq i \leq n$, every element of S is sent to 0.

Example 3. Let $(A, \sigma) = (\mathbb{Q}, \text{id})$ and $(B, \sigma) = (\mathbb{Q}^{\mathbb{N}}, \sigma)$, where σ is the shift (to the left) operator. Then the tuple $\mathbf{b} = ((1, 0, 1, 0, \dots), (2017, 1, 0, 1, \dots)) \in B$ is a solution of the equations

$$\begin{cases} \sigma(x_1) + x_1 - 1 = 0, \\ \sigma(x_2) - x_1 = 0. \end{cases}$$

Definition 6. If (A, σ) is a difference ring and $S \subseteq A\{x_1, \dots, x_n\}$ and B is a non-negative integer, the B -th *transform* of S is the set $\{\sigma^B(f) \mid f \in S\}$. So, the 0-th transform of S is S . The B -th transform of a system of difference equations is defined similarly.

Example 4. The 2-nd transform of the system

$$\begin{cases} \sigma(x_1)^5 = x_1 + x_2^2 \\ x_3^3 + x_1 + 1 = 0 \end{cases}$$

is the system

$$\begin{cases} \sigma^3(x_1)^5 = \sigma^2(x_1) + \sigma^2(x_2)^2 \\ \sigma^2(x_3)^3 + \sigma^2(x_1) + 1 = 0. \end{cases}$$

Definition 7. A difference equation $f(x_1, \dots, x_n) = 0$ is said to be a *consequence* of a system of difference equations $S = 0$ if there exists a non-negative integer B such that f belongs to the polynomial ideal generated by the 0-th, ..., B -th transforms of S , that is

$$f \in (\sigma^i(f) \mid 0 \leq i \leq B, f \in S).$$

Example 5. Let $S = 0$ be the system

$$\begin{cases} f_1 = x_2\sigma(x_1) - x_1 - 1 = 0 \\ f_2 = \sigma(x_2) - x_2^2 = 0. \end{cases}$$

The equation $\sigma^2(x_1)x_2^2 - \sigma(x_1) - 1$ is a consequence of $S = 0$ with $B = 1$, because

$$\sigma(f_1) - \sigma^2(x_1)f_2 = \sigma(x_2\sigma(x_1) - x_1 - 1) - \sigma^2(x_1)(\sigma(x_2) - x_2^2) = \sigma^2(x_1)x_2^2 - \sigma(x_1) - 1.$$

3. MAIN RESULTS

For all $d \in \mathbb{Z}_{\geq 0}$ and $D \in \mathbb{Z}_{> 0}$ we define

$$B(d, D) = \begin{cases} D + 1 & \text{if } d = 0, \\ 2 + D^2 + \frac{D(D-1)(D-2)}{6} & \text{if } d = 1, \\ B(d-1, D) + D^{B(d-1, D)} & \text{if } d > 1. \end{cases}$$

3.1. Effective difference Nullstellensatz.

Theorem 1. *Let k be a difference field and $F = \{f_1 = 0, \dots, f_N = 0\}$ a system of difference equations, with $f_1, \dots, f_N \in k\{u_1, \dots, u_r\}$. We set*

$$h_i := \max_{j=1, \dots, N} \text{ord}_{u_i} f_j \quad \text{and} \quad H = h_1 + \dots + h_r + r.$$

Let $d(F)$ and $D(F)$ denote the dimension and the sum of the degrees of the components of the affine variety defined by F over k in the affine H -space, respectively. Then system F has a solution in a difference ring containing k if and only if the system consisting of the 0-th, \dots , $(B(d, D) - 1)$ -th transforms of F is consistent as a system of polynomial equations.

Corollary 1. *Let $h = \max_{i=1, \dots, r} h_i$. If $f_1, \dots, f_N \in \mathbb{C}\{u_1, \dots, u_r\}$, then system F has a solution in $\mathbb{C}^{\mathbb{Z}}$ if and only if there exist tuples $\mathbf{u}_1, \dots, \mathbf{u}_r \in \mathbb{C}^{B(d, D) + h - 1}$, where $\mathbf{u}_i := (u_{i,1}, \dots, u_{i, B(d, D) + h})$, such that*

$$f_i(u_{1,j}, \dots, u_{r,j}) = 0 \text{ for all } 1 \leq i \leq N, \quad 1 \leq j \leq B(d, D) + h - 1.$$

Remark 2. We do not prove an effective strong Nullstellensatz generalizing Corollary 1, because such a statement is false as shown in Section 5.

3.2. Effective elimination. We will introduce the notation that will be used in Theorem 2. Let $\mathbf{x} = (x_1, \dots, x_m)$ and $\mathbf{u} = (u_1, \dots, u_r)$ be two sets of unknowns. Consider a system $F = \{f_1 = 0, \dots, f_N = 0\}$ of difference equations, where $f_1, \dots, f_N \in k\{\mathbf{x}, \mathbf{u}\}$. We set

$$h_i := \max_{j=1, \dots, N} \text{ord}_{u_i} f_j \quad \text{and} \quad H = h_1 + \dots + h_r + r.$$

Let E be the field of fractions of $k\{\mathbf{x}\}$ and X denote the affine variety defined by $f_1 = \dots = f_N = 0$ over E . We denote the dimension and the sum of the degrees of the components of X by $d(F)$ and $D(F)$, respectively.

Theorem 2. *For all integers $d \geq 0$ and $D \geq 1$ and systems F in \mathbf{x} and \mathbf{u} with $d(\Sigma) = d$ and $D(\Sigma) = D$, there exists a non-zero difference equation $f(\mathbf{x}) = 0$ that is a consequence of system F if and only if the ideal generated by the 0-th, \dots , $(B(d, D) - 1)$ -th transforms of F contains a nonzero polynomial depending only on \mathbf{x} and their transforms.*

3.3. Consequences for computation. Theorem 1 and Corollary 1 reduce consistency questions for systems of difference equations to consistency questions (in algebraically closed fields) of polynomial systems in finitely many variables and Theorem 2 reduces the question of existence/finding a consequence in the \mathbf{x} variables of a system of difference equations in the variables \mathbf{x} and \mathbf{u} to a question about a polynomial ideal in a polynomial ring in finitely many variables. These algebraic problems are classical and have been computationally solved using, for example, Gröbner bases, triangular sets, numerical algebraic geometry, etc. For all of these methods, implementations exist in many computer algebra systems and independent software packages (see, for example, [8, 3, 32]).

4. NUMERICAL VALUES AND PRACTICAL EXAMPLES

In the following table, we compute $B(d, D) - 1$ for small d and D .

$d \setminus D$	1	2	3	4	5
0	1	2	3	4	5
1	2	5	11	21	36

Remark 3. Almost all examples of modeling phenomena in the sciences using polynomial difference equations that we have seen in the literature can be written as systems with the same number of equations as unknowns in such a way that none of the equations is a consequence of the others. The above table is applicable to elimination problems for such systems with n equations if the problem is to eliminate $\lceil n/2 \rceil$ unknowns or less, as such problems typically result in varieties X (see the notation of Section 3.2) of dimension 0 or 1.

Remark 4. One can significantly speed up checking if an elimination is possible by

- (1) applying the number of transforms that is in the bound
- (2) substituting random values into the variables that are not being eliminated.

Similarly to [17], for each number p , $0 < p < 1$, using the Schwartz-Zippel lemma [36, Proposition 98], we can find the range for the random substitution so that the probability of the elimination being possible if and only if the “substituted” system has no solutions is greater than p . So, this would give an efficient probabilistic test for the possibility of elimination.

Remark 5. Although there could be special tricks and methods for each of the examples below, our approach provides a general and fully automated procedure.

Example 6. Consider the May-Leonard model for 2-plant annual competition, scaled down from [30]:

$$\begin{cases} x_{n+1} = \frac{(1-b)x_n}{x_n + \alpha_1 y_n} + bx_n, \\ y_{n+1} = \frac{(1-b)y_n}{\alpha_2 x_n + y_n} + by_n, \end{cases}$$

which can be rewritten as

$$(1) \quad \begin{cases} (x + \alpha_1 y)\sigma(x) = (1-b)x + bx(x + \alpha_1 y), \\ (\alpha_2 x + y)\sigma(y) = (1-b)y + by(\alpha_2 x + y), \end{cases}$$

where $F = \mathbb{Q}(\alpha_1, \alpha_2, b)$, with σ acting as the identity on F . To verify whether y can be eliminated from (1), we then consider the affine variety X defined by (1) over the field $\mathbb{Q}(\alpha_1, \alpha_2, b, x, \sigma(x))$ with coordinates $y, \sigma(y)$. A computation shows that $d = 0$ and $D = 1$, and so $B(d, D) - 1 = 2 - 1 = 1$. A computation shows that it is not only sufficient but also necessary to apply this single transform to perform the elimination. So, our main result gives a sharp upper bound for this example.

Example 7. Consider the May-Leonard model for 3-plant annual competition [30]:

$$\begin{cases} x_{n+1} = \frac{(1-b)x_n}{x_n + \alpha_1 y_n + \beta_1 z_n} + bx_n, \\ y_{n+1} = \frac{(1-b)y_n}{\alpha_2 x_n + y_n + \beta_2 z_n} + by_n, \\ z_{n+1} = \frac{(1-b)z_n}{\alpha_3 x_n + \beta_3 y_n + z_n} + bz_n, \end{cases}$$

which can be rewritten as

$$(2) \quad \begin{cases} (x + \alpha_1 y + \beta_1 z)\sigma(x) = (1-b)x + bx(x + \alpha_1 y + \beta_1 z), \\ (\alpha_2 x + y + \beta_2 z)\sigma(y) = (1-b)y + by(\alpha_2 x + y + \beta_2 z), \\ (\alpha_3 x + \beta_3 y + z)\sigma(z) = (1-b)z + bz(\alpha_3 x + \beta_3 y + z), \end{cases}$$

where $F = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_3, \beta_3, b)$, with σ acting as the identity on F . To verify whether y and z can be eliminated from (2), we consider the affine variety X defined by (2) over the field $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_3, \beta_3, b, x, \sigma(x))$ with coordinates $y, \sigma(y), z, \sigma(z)$. A computation shows that $d = 1$ and $D = 3$, and so $B(d, D) - 1 = 11$. A computation shows that

- two prolongations are necessary and sufficient
- carrying out a computation with 11 transforms as described in Remark 4 to check if an elimination is possible does not take significantly more time than doing this with two transforms.

Example 8. Consider the stage structured Leslie-Gower model [9, eq. (5)]:

$$\begin{cases} J_{n+1} = b_1 \frac{1}{1+d_1 A_n} A_n \\ A_{n+1} = s_1 \frac{1}{1+J_n+c_1 j_n} J_n \\ j_{n+1} = b_2 \frac{1}{1+d_2 a_n} a_n \\ a_{n+1} = s_2 \frac{1}{1+c_2 J_n+j_n} j_n, \end{cases}$$

which can be rewritten as

$$(3) \quad \begin{cases} (1 + d_1 A)\sigma(J) = b_1 A \\ (1 + J + c_1 j)\sigma(A) = s_1 J \\ (1 + d_2 a)\sigma(j) = b_2 a \\ (1 + c_2 J + j)\sigma(a) = s_2 j, \end{cases}$$

where $F = \mathbb{Q}(b_1, b_2, c_1, c_2, d_1, d_2, s_1, s_2)$ with σ acting as the identity on F . To verify whether J and j can be eliminated from (3), we consider the affine variety X defined by (3) over the field $\mathbb{Q}(b_1, b_2, c_1, c_2, d_1, d_2, s_1, s_2, a, A, \sigma(a), \sigma(A))$ with coordinates $j, \sigma(j), J, \sigma(J)$. A computation shows that $d = 0$; $D = 1$ as the equations are linear in $j, \sigma(j), J, \sigma(J)$. Then $B(d, D) - 1 = 2 - 1 = 1$. A computation shows that it is not only sufficient but also necessary to apply this single transform to perform the elimination. So, our main result gives a sharp upper bound for this example.

Example 9. A discrete multi-population SI model from [2], similarly to the previous examples, can be rewritten as

$$(4) \quad \begin{cases} \sigma(S) = S \left(1 - \frac{a \cdot \Delta t}{N_1} I - \frac{b \cdot \Delta t}{N_1} i \right) \\ \sigma(s) = s \left(1 - \frac{c \cdot \Delta t}{N_2} I - \frac{d \cdot \Delta t}{N_2} i \right) \\ \sigma(I) = I + S \left(\frac{a \cdot \Delta t}{N_1} I + \frac{b \cdot \Delta t}{N_1} i \right) \\ \sigma(i) = i + s \left(\frac{c \cdot \Delta t}{N_2} I + \frac{d \cdot \Delta t}{N_2} i \right), \end{cases}$$

where $F = \mathbb{Q}(a, b, c, d, \Delta t, N_1, N_2)$ with σ acting as the identity on F .

- To verify whether I, i can be eliminated from (4), we consider the affine variety defined by (4) over $\mathbb{Q}(a, b, c, d, \Delta t, N_1, N_2, s, \sigma(s), S, \sigma(S))$, and so $d = 0, D = 1$, thus $B(d, D) - 1 = 2 - 1 = 1$.
- To verify whether I, i, s can be eliminated from (4), we consider the affine variety defined by (4) over $\mathbb{Q}(a, b, c, d, \Delta t, N_1, N_2, S, \sigma(S))$, and so $d = 2, D = 2$. One can show that $B(2, 2) - 1 = 70 - 1 = 69$, which turned out to be computationally feasible as well.

Example 10. Let F_n be the n -th Fibonacci number. It turns out [11, p. 856] that the sequence $A_n := F_{2^n}$ satisfies a nonlinear difference equation. Such an equation can be found using difference elimination as follows. We introduce $B_n := F_{2^{n+1}}$. Then standard identities $F_{2k} = F_k(2F_{k+1} - F_k)$ and $F_{2k+1} = F_{k+1}^2 + F_k^2$ for the Fibonacci numbers imply the following system of difference equations

$$(5) \quad \begin{cases} A_{n+1} = A_n(2B_n - A_n), \\ B_{n+1} = A_n^2 + B_n^2. \end{cases}$$

Considered as a system of polynomial equations in B_n and B_{n+1} , (5) defines an affine variety of dimension zero and degree two over $\mathbb{Q}(A_n, A_{n+1})$. Theorem 2 implies that it is sufficient to consider system (5) and two of its transforms to eliminate B . Performing this elimination, we find the difference equation

$$5F_{2^n}^4 F_{2^{n+1}} - 2F_{2^n}^2 F_{2^{n+2}} + F_{2^{n+1}}^3 = 0,$$

giving an alternative to the difference equation stated in [11, p. 856].

Example 11. The following example shows that our bound is sharp in the case $d = 0$ (this is the case in Examples 6, 8, and 10). We fix a positive integer D and consider the system

$$(6) \quad \begin{cases} x(x-1) \cdots (x-D+1) = 0, \\ \sigma(x) - x - 1 = 0. \end{cases}$$

System (6) does not have a solution in $\mathbb{C}^{\mathbb{Z}}$, because the elements of the solution can only take values from $0, 1, \dots, D-1$ and strictly increase. On the other hand, the system consisting of the 0-th, \dots , $D-1 = (B(0, D) - 2)$ -th transforms of (6) has a solution $\sigma^i(x) = i$ for $0 \leq i \leq D$. Hence, it is necessary to consider one more transform in order to express 1 (i.e. eliminate x).

Example 12. The following example is based on an example constructed by E. Amzallag and R. Gustavson. Consider the system

$$(7) \quad \begin{cases} f_1 := x\sigma(x) = 0, \\ f_2 := x - \sigma(x) - \sigma^2(x) + 1 = 0. \end{cases}$$

Let X be the affine variety defined by (7) in the affine 3-space with coordinates $x, \sigma(x), \sigma^2(x)$. We have $\dim X = 1$ and $\deg X = 2$. Therefore, $B(d, D) - 1 = 5$. A calculation shows that

$$1 \in (f_1, \dots, \sigma^5(f_1), f_2, \dots, \sigma^5(f_2)) \quad \text{but} \quad 1 \notin (f_1, \dots, \sigma^4(f_1), f_2, \dots, \sigma^4(f_2)).$$

Thus, this shows that our upper bound for $d = 1$ and $D = 2$ is sharp.

5. COUNTEREXAMPLES

5.1. Failure of the standard DEP method. Consider the system of difference equations given by any set of generators of the polynomial ideal $I := I_1 \cap I_2$ of the polynomial ring $\mathbb{Q}[x, \sigma(x), y, \sigma(y), z, w]$, where

$$I_1 := (\sigma(y)z - 1, x, \sigma(x) - y), \quad I_2 := (\sigma(x), \sigma(y) - 1, (y-1)z - 1, (x-1)w - 1).$$

We do not present the actual generators of I to size of this set. A computation shows that

$$1 \in (I, \sigma(I), \sigma^2(I), \sigma^3(I), \sigma^4(I)).$$

Therefore, by Proposition 1, the system has no solutions in any difference ring. One can also show that

$$(8) \quad I = (I, \sigma(I)) \cap \mathbb{Q}[x, \sigma(x), y, \sigma(y), z, w],$$

$$(9) \quad \sigma(I) = (I, \sigma(I)) \cap \mathbb{Q}[\sigma(x), \sigma^2(x), \sigma(y), \sigma^2(y), \sigma(z), \sigma(w)].$$

Most of the existing effective bounds for systems of ordinary differential and difference equations [4, 10, 19, 20, 28] use geometric axioms [6, 29] as sufficient conditions for the existence of a solution based on the system and its first prolongation (differential equations) or first transform (difference equations), which are summarized under the DEP method mentioned in the introduction. In our case, it is tempting to formulate an analogue of such conditions as:

Let Γ be the affine variety defined by the system and its first transform. If the projections of Γ onto the varieties defined by the system and by its first transform alone, respectively, are dominant, then the system is consistent.

However, this is false in the above example as we have shown, where Γ is the affine variety corresponding to the ideal $(I, \sigma(I))$ in the affine space with coordinates $x, \sigma(x), \sigma^2(x), y, \sigma(y), \sigma^2(y), z, \sigma(z), w, \sigma(w)$, and (the Zariski closures of) the projections are given by the intersections in (8) and (9).

5.2. Non-existence of coefficient-independent effective strong Nullstellensatz. A (non-effective) strong Nullstellensatz for systems of difference equations can be stated as follows. Let $f_1 = \dots = f_N = 0$ be a system of difference equations. If a difference polynomial f vanishes at all solutions of the system in $\mathbb{C}^{\mathbb{Z}}$, then there exists ℓ such that f belongs to the radical of the ideal generated by the 0-th, \dots , ℓ -th transforms of f_1, \dots, f_N .

The following example shows that there is no uniform upper bound for this ℓ in terms of the degree, order, and number of variables of f_1, \dots, f_N . For every positive integer M , consider

$$(10) \quad \begin{cases} f_1 = \sigma(x) - x - \frac{1}{M} = 0, \\ f_2 = x(y(x-1) - 1) = 0. \end{cases}$$

Let $f = y(x-1) - 1$ and $x = \{x_n\}_{-\infty}^{\infty}$ and $y = \{y_n\}_{-\infty}^{\infty}$ any solution of (10) in $\mathbb{C}^{\mathbb{Z}}$. If $y_k(x_k - 1) - 1 \neq 0$ for some k , then $x_k = 0$. Hence, $x_{k+M} = 1$, and so

$$x_{k+M}(y_{k+M}(x_{k+M} - 1) - 1) = -1.$$

Therefore, f vanishes at every solution of (10) in $\mathbb{C}^{\mathbb{Z}}$. On the other hand, we claim that f does not belong to the radical of the ideal generated by the 0-th, \dots , $(M-1)$ -th transforms of f_1 and f_2 . These transforms belong to the polynomial ring $\mathbb{C}[x, \dots, \sigma^M(x), y, \dots, \sigma^{M-1}(y)]$. Consider the substitution

$$\sigma^k(x) = \frac{k}{M} \text{ for every } 0 \leq k \leq M, \quad \sigma^k(y) = \frac{M}{k-M} \text{ for every } 1 \leq k \leq M-1, \quad y = 0.$$

A direct computation shows that the polynomials $f_1, \dots, \sigma^{M-1}(f_1), f_2, \dots, \sigma^{M-1}(f_2)$ vanish after this substitution, but f does not.

6. PROOFS OF THE MAIN RESULTS

6.1. Difference Nullstellensatz.

Definition 8. We say that a difference ring (A, σ) is *invertive* if $\sigma : A \rightarrow A$ is an automorphism.

Remark 6. For any difference ring (A, σ) , there is an inversive difference ring (A^{inv}, σ) and a map of difference ring $(A, \sigma) \rightarrow (A^{\text{inv}}, \sigma)$ that is universal for maps from (A, σ) to inversive difference rings.

Definition 9. Given a difference ring (A, σ) , the *ring of inversive difference polynomials* over A in the variables, $A\{x_1, \dots, x_n\}^*$, may be realized as the ordinary polynomial ring over A in the formal variables $\sigma^j(x_i)$, for $j \in \mathbb{Z}$ and $1 \leq i \leq n$, with σ extending the given endomorphism on A and

$$\sigma(\sigma^j(x_i)) = \sigma^{j+1}(x_i)$$

on the variables.

Remark 7. If (A, σ) is inversive, then so is $A\{x_1, \dots, x_n\}^*$.

Definition 10. Let $S \subset k\{x_1, \dots, x_n\}$ be a finite set of difference polynomials and $h = \max\{\text{ord } P \mid P \in S\}$. The set of n tuples $\mathbf{a}_1, \dots, \mathbf{a}_n \in k^{\ell+h}$, where $\mathbf{a}_i := (a_{i,0}, \dots, a_{i,N+h-1})$, is called a *partial solution of length ℓ* if, for every $P \in S$ and $0 \leq s \leq \ell - 1$, polynomial $\sigma^s(P)$ vanishes after the substitution

$$\sigma^i(x_j) = a_{j,i} \text{ for every } 1 \leq j \leq n, \quad 0 \leq i \leq \ell + h - 1.$$

Let K be an inversive difference field. Then the difference ring of sequences $K^{\mathbb{Z}}$ with respect to the shift automorphism can be endowed with a structure of a difference K -algebra via the embedding of difference rings $i_K: K \rightarrow K^{\mathbb{Z}}$ defined by

$$i_K(f) = (\dots, \sigma^{-1}(f), f, \sigma(f), \sigma^2(f), \dots) \text{ for } f \in K.$$

This can be similarly done for $K^{\mathbb{N}}$.

Proposition 1. *For all uncountable algebraically closed fields K and finite sets $S \subseteq K\{x_1, \dots, x_n\}$, the following statements are equivalent:*

1. S has a solution in $K^{\mathbb{Z}}$.
2. S has a solution in $K^{\mathbb{N}}$.
3. S has finite partial solutions of length ℓ for all $\ell \gg 0$.
4. The ideal $[S] := (\{\sigma^j(P) \mid P \in S, j \in \mathbb{N}\}) \subseteq K\{x_1, \dots, x_n\}$ does not contain 1.
5. The ideal $[S]^* := (\{\sigma^j(P) \mid P \in S, j \in \mathbb{Z}\}) \subseteq K\{x_1, \dots, x_n\}^*$ does not contain 1.
6. S has a solution in some difference K -algebra.

Proof. The implications $1 \implies 2$, $2 \implies 3$, and $6 \implies 4$ are straightforward.

We show that $3 \implies 4$. Assume that there exists an arbitrary long partial solution, but $1 \in [S]$. Then there is an expression of the form

$$(11) \quad 1 = \sum_{i=0}^{\ell} \sum_{P \in S} a_{i,P} \sigma^i(P),$$

where $a_{i,P} \in K\{x_1, \dots, x_n\}$. Let $h = \max\{\text{ord } P \mid P \in S\}$. Consider a partial solution of S of the length $\ell + h$ and plug it into the equality (11). Then the right-hand side will vanish, so we arrive at contradiction.

We show that $4 \implies 5$. Assume that $1 \in [S]^*$. We fix some representation of 1 as an element of $[S]^*$. Let N be the maximum number such that $\sigma^{-N}(x_i)$ occurs in the representation. Applying σ^N to the both sides of the representation, we obtain a representation of 1 as an element of $[S]$.

We show that $5 \implies 6$. Let $\pi: K\{x_1, \dots, x_n\} \rightarrow K\{x_1, \dots, x_n\}/[S]^*$ be the canonical surjection. Then $(\pi(x_1), \dots, \pi(x_n))$ is a solution of S in $K\{x_1, \dots, x_n\}/[S]^*$.

We show $5 \implies 1$. Let E be the inversive difference subfield of K generated by the coefficients of elements of S over the prime subfield of K . Since 1 does not belong to $[S]^* \cap E\{x_1, \dots, x_n\}^*$, there exists a maximal (not necessarily difference) ideal $\mathfrak{m} \subset E\{x_1, \dots, x_n\}^*$ containing $[S]^* \cap E\{x_1, \dots, x_n\}^*$. Then $L := E\{x_1, \dots, x_n\}^* / \mathfrak{m}$ is a field, and the transcendence degree of L over E is at most countable. Since K is algebraically closed and has an uncountable transcendence degree, there exists an embedding $\varphi: L \rightarrow K$ over the common subfield E . Composing φ with the canonical surjection $E\{x_1, \dots, x_n\}^* \rightarrow L$, we obtain an E -algebra homomorphism $\psi: E\{x_1, \dots, x_n\} \rightarrow K$ such that $[S]^* \subset \text{Ker } \psi$. For every $1 \leq i \leq n$, we construct a sequence $\mathbf{a}_i := \{a_{i,j}\}_{j \in \mathbb{Z}} \in K^{\mathbb{Z}}$ by the formula

$$a_{i,j} = \psi(\sigma^j(x_i)).$$

A direct computation shows that $(\mathbf{a}_1, \dots, \mathbf{a}_n)$ is a solution of S in $K^{\mathbb{Z}}$. \square

6.2. Variety and two projections. Let k be a difference field and

$$F = \{f_1 = 0, \dots, f_N = 0\}$$

a system of difference equations, with $f_1, \dots, f_N \in k\{u_1, \dots, u_r\}$. We set

$$h_i := \max_{j=1, \dots, N} \text{ord}_{u_i} f_j \quad \text{and} \quad H = h_1 + \dots + h_r + r.$$

For the rest of Section 6, we fix K be an inversive algebraically closed difference field of uncountable transcendence degree containing k . With the system F of difference equations, we associate the following geometric data:

- the variety X defined by the polynomials f_1, \dots, f_N in \mathbb{A}^H ;
- two projections $\pi_1, \pi_2: \mathbb{A}^H \rightarrow \mathbb{A}^{H-r}$ defined by

$$\begin{aligned} \pi_1(u_1, \dots, \sigma^{h_1}(u_1), u_2, \dots, \sigma^{h_r}(u_r)) &:= (u_1, \dots, \sigma^{h_1-1}(u_1), u_2, \dots, \sigma^{h_r-1}(u_r)), \\ \pi_2(u_1, \dots, \sigma^{h_1}(u_1), u_2, \dots, \sigma^{h_r}(u_r)) &:= (\sigma(u_1), \dots, \sigma^{h_1}(u_1), \sigma(u_2), \dots, \sigma^{h_r}(u_r)). \end{aligned}$$

Let $Z \subset \mathbb{A}^H$ be a variety defined by polynomials $g_1, \dots, g_s \in K[V]$. Let $\sigma^i(Z)$, where $i \in \mathbb{Z}$, denote the variety defined by the polynomials $g_1^{\sigma^i}, \dots, g_s^{\sigma^i} \in K[V]$, where g^{σ^i} means the result of applying σ^i to all coefficients of g . The coordinate-wise application of σ^i defines a bijection between Z and $\sigma^i(Z)$.

Definition 11. A sequence of points $p_1, \dots, p_\ell \in \mathbb{A}^H(K)$ is said to be a *partial solution* of the triple (X, π_1, π_2) if

$$\begin{cases} \pi_1(p_{i+1}) = \pi_2(p_i) \text{ for every } 1 \leq i < \ell, \\ p_i \in \sigma^{i-1}(X)(K) \text{ for every } 1 \leq i \leq \ell. \end{cases}$$

A two-sided infinite sequence with such a property is called a *solution* of the triple (X, π_1, π_2) .

Lemma 1. *For every positive integer ℓ , system F has a partial solution of length ℓ if and only if the triple (X, π_1, π_2) has a partial solution of length ℓ .*

System F has a solution in $K^{\mathbb{Z}}$ if and only if the triple (X, π_1, π_2) has an infinite solution.

Proof. Let $h = \max_{1 \leq i \leq r} h_i$. Consider a partial solution $\mathbf{u}_1, \dots, \mathbf{u}_r \in K^{\ell+h}$ of F , where $\mathbf{u}_i = (u_{i,1}, \dots, u_{i,\ell+h})$ for every $1 \leq i \leq r$. We set

$$p_j := (u_{1,j}, \dots, u_{1,j+h_1}, u_{2,j}, \dots, u_{r,j+h_r}) \text{ for every } 1 \leq j \leq \ell.$$

By the construction

$$\pi_2(p_j) = (u_{1,j+1}, \dots, u_{1,j+h_1}, u_{2,j+1}, \dots, u_{r,j+h_r}) = \pi_1(p_{j+1}),$$

so $p_{j+1} \in \pi_1^{-1}(\pi_2(p_j))$ for every $1 \leq j \leq \ell - 1$. The definition of partial solution implies that $p_j \in \sigma^{j-1}(X)$ for every $1 \leq j \leq \ell$. Hence, p_1, \dots, p_ℓ is a partial solution of the triple (X, π_1, π_2) . The above argument can be straightforwardly reversed to construct a partial solution of F from a partial solution of (X, π_1, π_2) . The case of infinite solutions is completely analogous. \square

In the introduced geometric language, we can formulate the following question equivalent to effective difference Nullstellensatz

Question. *Let X be an algebraic subvariety of \mathbb{A}^H and π_1, π_2 be surjective linear maps $\mathbb{A}^H \rightarrow \mathbb{A}^{H-r}$. How long a partial solution of (X, π_1, π_2) is it sufficient to find in order to conclude that the triple (X, π_1, π_2) has an infinite solution?*

Thus, in what follows, we fix a triple (X, π_1, π_2) , where X is an algebraic variety of \mathbb{A}^H and π_1, π_2 are surjective linear maps $\mathbb{A}^H \rightarrow \mathbb{A}^{H-r}$ defined over the σ -constants of K .

6.2.1. *Trains.* The goal of this section is to generalize the notion of a solution of the triple to not necessarily zero-dimensional points.

Definition 12. For two irreducible subvarieties $Y_1, Y_2 \subset V$, we say that the generic point of Y_1 maps to the generic point of Y_2 (and denote it by $Y_1 \mapsto Y_2$) if $\overline{\pi_1(Y_2)} = \overline{\pi_2(Y_1)}$.

Definition 13. For ℓ a positive integer or $+\infty$, a sequence of irreducible subvarieties (Y_1, \dots, Y_ℓ) in V is said to be a *train* of length ℓ in X if

$$\begin{cases} Y_i \mapsto Y_{i+1} \text{ for every } 1 \leq i < \ell, \\ Y_i \subset \sigma^{i-1}(X) \text{ for every } 1 \leq i \leq \ell. \end{cases}$$

Remark 8. Every partial solution can be considered a train of zero-dimensional subvarieties.

Lemma 2. *For every train (Y_1, \dots, Y_ℓ) in X , there exists a partial solution p_1, \dots, p_ℓ of (X, π_1, π_2) such that, for all i , $1 \leq i \leq \ell$, we have $p_i \in Y_i$.*

Proof. We will prove the following statement by induction on ℓ : there exists a nonempty open subset $U \subset Y_\ell$ such that, for every point $p_\ell \in U$, there exists a partial solution p_1, \dots, p_ℓ of (X, π_1, π_2) such that, for every i , $1 \leq i \leq \ell$, we have $p_i \in Y_i$. In the case $\ell = 1$, we can set $U = Y_1$, because every single point in X is a partial solution of length one.

Assume that $\ell > 1$. Applying the inductive hypothesis to the train $(Y_1, \dots, Y_{\ell-1})$, we obtain an open nonempty subset $U_0 \subset Y_{\ell-1}$. Since U_0 is dense in $Y_{\ell-1}$, $\pi_2(U_0)$ is dense in $\overline{\pi_2(Y_{\ell-1})} = \overline{\pi_1(Y_\ell)}$. Since $\pi_1(Y_\ell)$ is a constructible dense subset in $\overline{\pi_1(Y_\ell)}$, $\pi_2(U_0) \cap \pi_1(Y_\ell)$ is also a dense constructible in $\overline{\pi_1(Y_\ell)}$. Let $U_1 \subset \pi_2(U_0) \cap \pi_1(Y_\ell)$ be an open dense subset of $\overline{\pi_1(Y_\ell)}$. Then $U_2 := Y_\ell \cap \pi_1^{-1}(U_1)$ is nonempty open in Y_ℓ . We claim that every point $p_\ell \in U_2$ can be extended to a partial solution p_1, \dots, p_ℓ such that $p_i \in Y_i$. By the definition of U_2 , $\pi_1(p_\ell) \in \pi_2(U_0)$, so there exists $p_{\ell-1} \in U_0$ such that $\pi_2(p_{\ell-1}) = \pi_1(p_\ell)$. By the inductive hypothesis, $p_{\ell-1}$ can be further extended to a partial solution. \square

Corollary 2. *If there exists an infinite train in X , then there exists a solution for the triple (X, π_1, π_2) .*

Proof. Since there exists an infinite train, then there exists an arbitrarily long train. Due to Lemma 2, there exists an arbitrarily long solution of (X, π_1, π_2) . Lemma 1 implies that there exists an arbitrarily long solution of the corresponding system F . Hence, due to Proposition 1, there exists a solution of F in $K^{\mathbb{Z}}$. Lemma 1 implies that there exists an infinite solution of the triple (X, π_1, π_2) . \square

Definition 14. For two trains Y and Y' of the same length, the inclusion $Y \subset Y'$ is understood as a component-wise containment.

For a train Y in X and $i \in \mathbb{Z}$, $\sigma^i(Y)$ is the result of the component-wise application of σ^i to Y , and, since π_1 and π_2 are defined over the constants, $\sigma^i(Y)$ is a train in $\sigma^i(X)$.

Remark 9. Since the component-wise union of any chain of trains of the same length is again a train of this length, trains of fixed length satisfy Zorn's lemma with respect to inclusion. Hence, maximal trains of a fixed length are well-defined.

6.2.2. The number of maximal trains.

Lemma 3. Let $\varphi_X: X \rightarrow Z$ and $\varphi_Y: Y \rightarrow Z$ be dominant morphisms of affine varieties over an algebraically closed field. Assume that X and Y are irreducible. Consider the fibered product $X \times_Z Y$ of φ_X and φ_Y , considered as a variety, and denote the natural morphisms to X and Y by π_X and π_Y , respectively. Then there exists an irreducible component $V \subset X \times_Z Y$ such that the restrictions of both π_X and π_Y to V are dominant.

Proof. Denote the algebras of regular functions on X , Y , and Z by A , B , and C , respectively. Since X , Y , and Z are irreducible (Z is irreducible as an image of an irreducible variety under a dominant morphism), these algebras are domains. We denote the fields of fractions of A , B , and C by E , F , and L , respectively. The dominant maps φ_X and φ_Y give rise to injective homomorphisms $\varphi_X^\#: C \rightarrow A$ and $\varphi_Y^\#: C \rightarrow B$. These homomorphisms equip A and B with a C -algebra structure. Then, the algebra of regular functions on $X \times_Z Y$, as a scheme, is $A \otimes_C B$ (see [1, Lemma 25.6.7]).

Let \mathfrak{p} be any prime ideal in $E \otimes_L F$. Let $D := (E \otimes_L F)/\mathfrak{p}$ and $\pi: E \otimes_L F \rightarrow D$ be the canonical projection. Consider the natural homomorphism $i: A \otimes_C B \rightarrow E \otimes_L F$. Since $1 \in i(A \otimes_C B)$, the composition $\pi \circ i$ is a nonzero homomorphism. Consider the natural embeddings $i_A: A \rightarrow A \otimes_C B$ and $i_B: B \rightarrow A \otimes_C B$. We will show that the compositions $\pi \circ i \circ i_A: A \rightarrow D$ and $\pi \circ i \circ i_B: B \rightarrow D$ are injective. Introducing the natural embeddings $i_E: E \rightarrow E \otimes_L F$ and $j_A: A \rightarrow E$, we can rewrite

$$\pi \circ i \circ i_A = \pi \circ i_E \circ j_A.$$

The homomorphisms i_E and j_A are injective. The restriction of π to $i_E(E)$ is also injective, since E is a field. Hence, the whole composition $\pi \circ i_E \circ j_A$ is injective. The argument for $\pi \circ i \circ i_B$ is analogous.

Thus, we have an irreducible subvariety of $X \times_Z Y$, and hence of the variety $(X \times_Z Y)_{\text{red}}$ [1, Lemma 25.12.6], defined by the ideal $\text{Ker}(\pi \circ i)$ that projects dominantly on both X and Y . Hence, the component containing this subvariety also projects dominantly on X and Y . \square

Definition 15. Let $X = X_1 \cup X_2 \cup \dots \cup X_s$ be the decomposition of X into irreducible components. A pair (Y, \mathbf{c}) , where $Y = (Y_1, \dots, Y_\ell)$ is a train in X and $\mathbf{c} = (c_1, \dots, c_\ell) \in [1, \dots, s]^\ell$, is called a *marked train* of length ℓ and *signature* \mathbf{c} if $Y_i \subset \sigma^{i-1}(X_{c_i})$ for every $1 \leq i \leq \ell$.

For every train, one can assign a signature, so it becomes a marked train. However, this signature might be not unique. Analogously to trains, we define a notion of a maximal train of given length ℓ and signature \mathbf{c} . Let $\mathbf{c} = (c_1, \dots, c_\ell) \in [1, \dots, s]^\ell$ be a tuple. Consider

$$X^{\mathbf{c}} := X_{c_1} \times \sigma(X_{c_2}) \times \dots \times \sigma^{\ell-1}(X_{c_\ell}) \subset (\mathbb{A}^H)^\ell.$$

We denote the projections $(\mathbb{A}^H)^\ell \rightarrow \mathbb{A}^H$ onto the components by $\psi_{\ell,1}, \dots, \psi_{\ell,\ell}$, respectively. We introduce

$$(12) \quad W_{\mathbf{c}} := \{p \in X^{\mathbf{c}} \mid \pi_2(\psi_{\ell,i}(p)) = \pi_1(\psi_{\ell,i+1}(p)) \text{ for all } i, 1 \leq i < \ell\}.$$

The restrictions of $\psi_{\ell,1}, \dots, \psi_{\ell,\ell}$ to $W_{\mathbf{c}}$ will be denoted by the same symbols.

Lemma 4. *For every irreducible subvariety $Y \subset W_{\mathbf{c}}$,*

$$\left(\overline{\psi_{\ell,1}(Y)}, \dots, \overline{\psi_{\ell,\ell}(Y)}\right)$$

is a marked train of signature \mathbf{c} .

Proof. For every $i, 1 \leq i \leq \ell$,

$$Y_i := \overline{\psi_{\ell,i}(Y)} \subset \overline{\psi_{\ell,i}(W_{\mathbf{c}})} \subset \sigma^{i-1}(X_{c_i}).$$

Moreover, since Y is irreducible, $\overline{\psi_{\ell,i}(Y)}$ is also irreducible. Fix some $i, 1 \leq i < \ell$. We will show that $\overline{\pi_2(Y_i)} = \overline{\pi_1(Y_{i+1})}$. We can write $\overline{\pi_2(Y_i)}$ as $\overline{\pi_2(\psi_{\ell,i}(Y))}$. By (12), the latter is equal to $\overline{\pi_1(\psi_{\ell,i+1}(Y))}$, which is the same as $\overline{\pi_1(Y_{i+1})}$. \square

Lemma 5. *For every marked train (Y_1, \dots, Y_ℓ) of signature \mathbf{c} in X , there exists an irreducible subvariety $Y \subset W_{\mathbf{c}}$ such that, for every $i, 1 \leq i \leq \ell$, we have $Y_i = \overline{\psi_{\ell,i}(Y)}$.*

Proof. We will prove the lemma by induction on ℓ . For $\ell = 1$, $\mathbf{c} = (c_1)$, $W_{\mathbf{c}} = X_{c_1}$, and we can set $Y = Y_1$.

Let $\ell > 1$. We apply the inductive hypothesis to the train $(Y_1, \dots, Y_{\ell-1})$ of signature $\mathbf{c}' := (c_1, \dots, c_{\ell-1})$ and obtain an irreducible subvariety $Y' \subset W_{\mathbf{c}'} \subset V^{\ell-1}$. Then there is a natural embedding of $Y' \times Y_\ell$ into V^ℓ . Denote $(Y' \times Y_\ell) \cap W_{\mathbf{c}}$ by W . Since Y' is already contained in $W_{\mathbf{c}'}$,

$$(13) \quad W = \{p \in Y' \times Y_\ell \mid \pi_2(\psi_{\ell,\ell-1}(p)) = \pi_1(\psi_{\ell,\ell}(p))\}.$$

Denote the projection of V^ℓ to the first $H(\ell - 1)$ coordinates by ψ , and let

$$(14) \quad Z := \overline{\pi_2(\psi_{\ell-1,\ell-1}(Y'))} = \overline{\pi_1(Y_\ell)}.$$

Then equality (13) implies (see [16, Ex. 2.26]) that W together with the morphisms $\psi: W \rightarrow Y'$ and $\psi_{\ell,\ell}: W \rightarrow Y_\ell$ is the fibered product of the morphisms $\pi_2 \circ \psi_{\ell-1,\ell-1}: Y' \rightarrow Z$ and $\pi_1: Y_\ell \rightarrow Z$. Equality (14) implies that both of these morphisms are dominant.

Due to Lemma 3, there exists an irreducible subset $Y \subset W$ such that $\psi: Y \rightarrow Y'$ and $\psi_{\ell,\ell}: Y \rightarrow Y_\ell$ are dominant. For every $i, 1 \leq i < \ell$, since $\psi_{\ell,i} = \psi_{\ell-1,i} \circ \psi$, the restriction $\psi_{\ell,i}: Y \rightarrow Y_i$ is dominant as a composition of two dominant morphisms. \square

Lemma 6. *Let the degree of X_i be D_i , and fix a tuple $\mathbf{c} = (c_1, \dots, c_\ell) \in [1, \dots, s]^\ell$. The number of maximal trains of signature \mathbf{c} in X does not exceed $D_{c_1} \cdot D_{c_2} \cdot \dots \cdot D_{c_\ell}$.*

Proof. Since $W_{\mathbf{c}}$ is the intersection of $X^{\mathbf{c}}$ with a linear subspace,

$$(15) \quad \deg W_{\mathbf{c}} \leq \deg X^{\mathbf{c}} = \deg X_{c_1} \cdot \deg \sigma(X_{c_2}) \cdot \dots \cdot \deg \sigma^{\ell-1}(X_{c_\ell}).$$

Since application of σ to a variety does not change the degree, the product in (15) does not exceed $D_{c_1} \cdot \dots \cdot D_{c_\ell}$. Hence, the number of components of $W_{\mathbf{c}}$ does not exceed $D_{c_1} \cdot \dots \cdot D_{c_\ell}$.

Let (Y_1, \dots, Y_ℓ) be a maximal train in X of signature \mathbf{c} . Lemma 5 implies that there exists an irreducible subvariety $Y \subset W_{\mathbf{c}}$ such that for every $i, 1 \leq i \leq \ell$, we have $Y_i = \overline{\psi_{\ell,i}(Y)}$. Let C be an irreducible component of $W_{\mathbf{c}}$ containing Y . Lemma 4 implies that

$$\left(\overline{\psi_{\ell,1}(C)}, \dots, \overline{\psi_{\ell,\ell}(C)}\right)$$

is also a train of signature \mathbf{c} . Moreover, since $C \supset Y$, this train contains (Y_1, \dots, Y_ℓ) . The maximality of the latter implies that these trains are equal. Hence, Y could be chosen to be an irreducible component of $W_{\mathbf{c}}$. Thus, we obtain an injective map from the set of maximal trains of signature \mathbf{c} to the set of all irreducible component of $W_{\mathbf{c}}$. Hence, the number of maximal trains also does not exceed $D_{c_1} \cdot \dots \cdot D_{c_\ell}$. \square

Corollary 3. *Let $D = \sum_{i=1}^s D_i$ be the sum of the degrees of the components of X . Then the number of maximal trains in X of length ℓ does not exceed D^ℓ .*

Proof. Since every maximal train can be considered as a marked maximal train, the number of maximal trains of length ℓ in X does not exceed the sum of products $D_{c_1} \cdot \dots \cdot D_{c_\ell}$ over all tuples \mathbf{c} of length ℓ . This sum is equal to

$$\sum_{c_1=1}^s \sum_{c_2=1}^s \dots \sum_{c_\ell=1}^s \prod_{i=1}^{\ell} D_{c_i} = (D_1 + \dots + D_s)^\ell = D^\ell. \quad \square$$

6.2.3. A bound for trains.

Definition 16. For a train $Y = (Y_1, \dots, Y_\ell)$ in X , we introduce the codimension of Y as

$$\text{codim } Y := \dim X - \min_{1 \leq i \leq \ell} \dim Y_i.$$

Definition 17. We define $B'(d, D)$ to be the smallest natural number N such that, for every triple (X, π_1, π_2) such that the sum of the degrees of the components of X does not exceed D , the existence of a train of length N and codimension at most d in X implies the existence of an infinite train in X , or ∞ if such N does not exist.

The following statement implies that $B'(d, D)$ is finite for all $d \in \mathbb{Z}_{\geq 0}$ and $D \in \mathbb{Z}_{> 0}$ and gives an upper bound for $B'(d, D)$.

Proposition 2. *For all $D \in \mathbb{Z}_{> 0}$,*

- (1) $B'(0, D) \leq D + 1$ and
- (2) for every $d \in \mathbb{Z}_{\geq 0}$, $B'(d + 1, D) \leq B'(d, D) + D^{B'(d, D)}$.

Proof. Throughout the proof, we will use the observation that the existence of an infinite train in $\sigma^i(X)$ for some $i \in \mathbb{Z}$ implies (via component-wise application of σ^{-i}) the existence of an infinite train of the same codimension in X .

We will also use the following construction for obtaining infinite trains. Consider a train (Y_1, \dots, Y_ℓ) of length $\ell > 1$. If $Y_\ell = \sigma^{\ell-1}(Y_1)$, then we can construct an infinite train as follows:

$$(16) \quad (Y_1, Y_2, \dots, Y_{\ell-1}, \sigma^{\ell-1}(Y_1), \sigma^{\ell-1}(Y_2), \dots, \sigma^{\ell-1}(Y_{\ell-1}), \sigma^{2\ell-1}(Y_1), \dots).$$

We prove the first statement of the proposition. Consider a train (Y_1, \dots, Y_{D+1}) of codimension zero and length $D + 1$. Since, for every i , $1 \leq i \leq D + 1$, we have $\dim Y_i = \dim X$, then every $\sigma^{-i+1}(Y_i)$ is an irreducible component of X . The number of components of X does not exceed D , so some of the $\sigma^{-i+1}(Y_i)$'s coincide. If $\sigma^{-i+1}(Y_i) = \sigma^{-j+1}(Y_j)$ for some $i < j$, then $Y_j = \sigma^{j-i}(Y_i)$, so we can construct an infinite train of codimension zero, as in (16).

We prove the second statement of the proposition. Consider a train (Y_1, \dots, Y_B) of codimension at most $d + 1$ and length $B := B'(d, D) + D^{B'(d, D)}$. We introduce $N := D^{B'(d, D)} + 1$

trains $Z^{(1)}, \dots, Z^{(N)}$ of length $\ell := B'(d, D)$ in $X, \sigma(X), \dots, \sigma^{N-1}(X)$, respectively, such that, for all $i, 1 \leq i \leq N$, we have

$$Z^{(i)} = \left(Z_1^{(i)}, \dots, Z_\ell^{(i)} \right) := (Y_i, \dots, Y_{i+\ell-1}).$$

For every $i, 1 \leq i \leq N$, consider a maximal train $\tilde{Z}^{(i)} = (\tilde{Z}_1^{(i)}, \dots, \tilde{Z}_\ell^{(i)})$ of length ℓ in $\sigma^{i-1}(X)$ containing $Z^{(i)}$. Then $\sigma^{-i+1}(\tilde{Z}^{(i)})$ is a maximal train of length ℓ in X . If there exists i such that $\text{codim } \tilde{Z}^{(i)} \leq d$, then there is an infinite train of codimension at most d due to the definition of $B'(d, D)$. Otherwise, $\text{codim } \tilde{Z}^{(i)} = d + 1$ for every $1 \leq i \leq N$.

Corollary 3 implies that there are at most $D^\ell = N - 1$ maximal trains of length ℓ in X . Hence, there are a and $b, 1 \leq a < b \leq N$, such that

$$\sigma^{-a+1}(\tilde{Z}^{(a)}) = \sigma^{-b+1}(\tilde{Z}^{(b)}).$$

Since $\text{codim } \tilde{Z}^{(a)} = \text{codim } \tilde{Z}^{(b)} = d + 1$, there exists $j, 1 \leq j \leq \ell$, such that

$$\dim \tilde{Z}_j^{(a)} = \dim \tilde{Z}_j^{(b)} = \dim X - (d + 1).$$

Hence, since both $\tilde{Z}_j^{(a)}$ and $Z_j^{(a)}$ are irreducible, $\dim Z_j^{(a)} \geq \dim X - (d + 1)$ and $Z_j^{(a)} \subset \tilde{Z}_j^{(a)}$, they are equal. Analogously, $\tilde{Z}_j^{(b)} = Z_j^{(b)}$. Therefore,

$$\sigma^{-a+1}(Y_{a+j-1}) = \sigma^{-a+1}(Z_j^{(a)}) = \sigma^{-a+1}(\tilde{Z}_j^{(a)}) = \sigma^{-b+1}(\tilde{Z}_j^{(b)}) = \sigma^{-b+1}(Z_j^{(b)}) = \sigma^{-b+1}(Y_{b+j-1}).$$

Hence,

$$Y_{b+j-1} = \sigma^{b-a}(Y_{a+j-1}),$$

so, as in (16), there exists an infinite train of codimension at most $d + 1$. \square

The proof of Proposition 2 implies.

Corollary 4. *If (X, π_1, π_2) has an arbitrarily long partial solution, that there exists an infinite skew-cyclic train in X , i.e. a train of the form*

$$(\dots, Y_1, Y_2, \dots, Y_\ell, \sigma^\ell(Y_1), \sigma^\ell(Y_2), \dots),$$

where $Y_i \in \sigma^{i-1}(X)$ for $1 \leq i \leq \ell$.

Proposition 3. $B'(1, D) \leq 2 + D^2 + \frac{D(D-1)(D-2)}{6}$ for every $D \geq 1$.

Proof. Assume that there is no infinite train of codimension at most one in X . Let

$$X = X_1 \cup X_2 \cup \dots \cup X_s$$

be the irreducible decomposition of X and $D_i := \deg X_i$. We construct a directed graph (with loops and multiple edges) G with vertices numbered from 1 to s as follows. For every maximal train among the marked trains of signature (i, j) in X , we draw an edge from i to j (the number of such trains is finite by Lemma 6). The codimension of an edge is defined to be the codimension of the corresponding train.

Assume that there is a directed cycle (c_1, \dots, c_ℓ) consisting of edges of codimension zero. Then there is a train

$$(X_{c_1}, \sigma(X_{c_2}), \dots, \sigma^{\ell-1}(X_{c_\ell}), \sigma^\ell(X_{c_1})),$$

so, as in (16), there exists an infinite train of codimension zero in X . Hence, in what follows, we will assume that there is no such a directed cycle in G . Therefore, we can reenumerate the components in such a way that $i < j$ for every codimension zero edge (i, j) .

Consider a train $Y = (Y_1, \dots, Y_\ell)$ of codimension one in X . The train Y can be considered as a marked train with respect to a signature $\mathbf{c} = (c_1, \dots, c_\ell)$. For every $1 \leq i < \ell$, we assign an edge e_i in G corresponding to any maximal marked train of signature (c_i, c_{i+1}) containing $(\sigma^{-i+1}(Y_i), \sigma^{-i+1}(Y_{i+1}))$. Hence, there exists a path

$$(e_1, \dots, e_{\ell-1})$$

in G that corresponds to the whole train Y . Assume that some edge e corresponding to a maximal train (Z_1, Z_2) of codimension one occurs twice in this path, so $e = e_i = e_j$ for some $1 \leq i < j < \ell$. Without loss of generality, we may assume that

$$\dim Z_1 = \dim X - 1.$$

Since $\dim Y_i$ and $\dim Y_j$ are both at least $\dim X - 1$ and (Z_1, Z_2) is maximal, we conclude that

$$Z_1 = \sigma^{-i+1}(Y_i) = \sigma^{-j+1}(Y_j).$$

Hence, as in (16), there exists an infinite train of codimension one in X . Thus, in what follows, we will assume that every edge of codimension one occurs in the path $(e_1, \dots, e_{\ell-1})$ at most once.

For an edge $e = (i, j)$, we introduce the weight $w(i, j) := i - j$. Let

$$N_+ := |\{i \mid w(e_i) \geq 0\}| \text{ and } N_- := |\{i \mid w(e_i) < 0\}|$$

and

$$W_+ := \sum_{i=1}^{\ell-1} \max\{0, w(e_i)\} \text{ and } W_- := \sum_{i=1}^{\ell-1} \min\{0, w(e_i)\}.$$

By the above reenumeration, all edges with positive weight are of codimension at least one. Therefore, N_+ does not exceed the number of maximal marked trains with signatures of the form (i, j) with $i \geq j$. Hence, due to Lemma 6, we obtain

$$N_+ \leq \sum_{i \geq j} D_i D_j.$$

Since the sum of weights along any path between vertices a and b is equal to $a - b$,

$$W_+ + W_- \geq -s + 1.$$

Combining this inequality with the fact that $N_- \leq -W_-$, we obtain

$$N_- \leq W_+ + s - 1.$$

Due to Lemma 6,

$$W_+ \leq \sum_{i \geq j} (i - j) D_i D_j.$$

Thus,

$$(17) \quad \ell - 1 = N_+ + N_- \leq \sum_{i \geq j} D_i D_j + \sum_{i \geq j} (i - j) D_i D_j + s - 1 \leq D^2 + \sum_{i \geq j} (i - j - 1) D_i D_j + s - 1.$$

For all integers $q, z_1, \dots, z_q \geq 1$, let

$$f_q(z_1, \dots, z_q) := \sum_{1 \leq j < i \leq q} (i - j - 1) z_i z_j + q - 1.$$

We claim that, if the sum

$$z_1 + \dots + z_q = \hat{D} := D_1 + \dots + D_s \leq D$$

is fixed, then the maximal value of f is achieved at

$$q = \hat{D} \iff z_1 = z_2 = \dots = z_q = 1.$$

To prove the claim, consider any other integer $p \geq 1$ and a tuple of positive integers (w_1, \dots, w_p) . Let $r \leq p$ be the largest integer such that $w_r \neq 1$. We have

$$\begin{aligned} & f_{p+1}(w_1, \dots, w_r - 1, \dots, w_p, 1) - f_p(w_1, \dots, w_p) = \\ &= 1 + \sum_{i \neq r} w_i(p - i - (|r - i| - 1)) + (w_r - 1)(p - r) = \\ &= 1 + \sum_{i < r} w_i(p - i - r + i + 1) + \sum_{i > r} (p - i + r - i + 1) + (w_r - 1)(p - r) = \\ &= 1 + \sum_{i < r} w_i(p - r + 1) + (p + r + 1)(p - r) - \sum_{i > r} 2i + (w_r - 1)(p - r) = \\ &= 1 + \sum_{i < r} w_i(p - r + 1) + (w_r - 1)(p - r) > 0. \end{aligned}$$

Hence, the claim is proved. Combining the claim with (17), we obtain

$$\ell \leq 1 + D^2 + \sum_{1 \leq j < i \leq D} (i - j - 1) = 1 + D^2 + \sum_{i=1}^{D-2} i(D - 1 - i) = 1 + D^2 + \frac{D(D-1)(D-2)}{6}. \quad \square$$

Propositions 2 and 3 imply

Corollary 5. *For all $d \in \mathbb{Z}_{\geq 0}$ and $D \in \mathbb{Z}_{> 0}$, $B'(d, D) \leq B(d, D)$.*

6.3. Proof of effective Nullstellensatz.

Proof of Theorem 1. We will use the notation introduced in Section 6.2. The fact that the system consisting of 0-th, \dots , $B(d, D) - 1$ -th transforms of F considered as a polynomial system is consistent implies that F has a partial solution of length $B(d, D) \geq B'(d, D)$. Lemma 1 implies that there is a partial solution of the triple (X, π_1, π_2) of length $B'(d, D)$. This partial solution is a train in X of codimension $\dim X = d$ and length $B'(d, D)$. Definition of $B'(d, D)$ implies that there exists an infinite train in X . Then Lemma 1 and Corollary 2 imply that system F has a solution in some difference ring extending k . \square

Proof of Corollary 1. We will use the notation introduced in Section 6.2. If $k = \mathbb{C}$, then K can be chosen to be \mathbb{C} , too. The statement of the corollary implies that system F has a partial solution of length $B(d, D)$. Analogously to the proof of Theorem 1, we have that system F is consistent. Then Proposition 1 implies that F has a solution in $\mathbb{C}^{\mathbb{Z}}$. \square

6.4. Proof of effective elimination.

Proof of Theorem 2. Let $E_0 \supset E$ be any difference field extension such that E_0 is algebraically closed and has an uncountable transcendence degree over the prime subfield. Since the difference ideal generated by Σ in $k\{\mathbf{x}, \mathbf{y}\}$ contains a nonzero polynomial depending only on \mathbf{x} and their transforms, the difference ideal generated by Σ in $E_0\{\mathbf{y}\}$ contains 1. So, the system does not have a solution in $E_0^{\mathbb{Z}}$. Theorem 1 implies that system Σ does not have a partial solution

in E_0 of length $B(d, D)$. Hence, the ideal generated by $B(d, D)$ transforms of Σ contains 1. Since the ideal is defined over E , there is an expression of 1 over E of the form

$$1 = \sum_{i=0}^{B(d,D)-1} \left(\sum_{j=1}^N c_{i,j} \sigma^i(g_j) + \sum_{j=1}^m c'_{i,j} \sigma^i(\sigma(y_j) - y_{k_j}) \right),$$

where $c_{i,j}, c'_{i,j} \in E$. Multiplying both sides of the above equality by the product of the denominators of $c_{i,j}$'s and $c'_{i,j}$'s, we obtain an expression of a nonzero polynomial from $k\{\mathbf{x}\}$ as an $k\{\mathbf{x}, \mathbf{y}\}$ -linear combination of $B(d, D)$ transforms of Σ . \square

7. DIFFERENCE NULLSTELLENSATZ OVER SMALL FIELDS

With our Proposition 1 we had required that the field from which we construct the sequence ring in which solutions to differences are sought be uncountable. In practice, this is a harmless assumption as one might take that field to be \mathbb{C} . However, this result may be conceptually unsatisfying and one might wish to find solutions to difference equations in sequences taken from a small field, such as the field of algebraic numbers. With the next proposition we show how to weaken the uncountability hypothesis by appealing to a more refined equivalent condition to the consistency of a system of difference equations coming from our work towards the effective Nullstellensatz and a theorem of Hrushovski on the limit theory of the Frobenius automorphisms [18]. Our invocation of Hrushovski's theorem is essentially contained in Fakhruddin's proof of the density of periodic points for polarized algebraic dynamical systems in [13]. For our purposes a slightly weaker result due to Varshavsky [35] suffices.

Proposition 4. *For all algebraically closed fields K (without any restriction on the cardinality) and finite sets $S \subseteq K\{x_1, \dots, x_n\}$, the following statements are equivalent:*

1. S has a solution in $K^{\mathbb{Z}}$.
2. S has a solution in $K^{\mathbb{N}}$.
3. S has finite partial solutions in K^N for all $N \gg 0$.
4. The ideal $[S] := (\{\sigma^j(P) \mid P \in S, j \in \mathbb{N}\}) \subseteq K\{x_1, \dots, x_n\}$ does not contain 1.
5. The ideal $[S]^* := (\{\sigma^j(P) \mid P \in S, j \in \mathbb{Z}\}) \subseteq K\{x_1, \dots, x_n\}^*$ does not contain 1.
6. S has a solution in some difference K -algebra.

In order to prove Proposition 4, we will extract two consequences of [35]. In Lemma 7 and 8, ϕ_s denotes the s -th power of the Frobenius automorphism.

Lemma 7. *Let R be a finitely generated difference subring of a difference field K . Then there exist a prime p , a positive integer s , and a difference homomorphism $\psi: R \rightarrow \mathbb{F}$, where \mathbb{F} is the algebraic closure of \mathbb{F}_p considered as a difference ring with respect to ϕ_s .*

Proof. Let R be generated by a_1, \dots, a_ℓ . Since R is a subring of a field, the ideal

$$(18) \quad I := \{f \in \mathbb{Z}\{X_1, \dots, X_\ell\} \mid f(a_1, \dots, a_\ell) = 0\}$$

is a prime difference ideal. As such, because every finitely generated difference ring is a Ritt difference ring [7, Chapter 3, Theorems II, and V], I is finitely generated as a perfect difference ideal. Let S be a finite set of such generators for I and h the largest order of the elements of S . Let b_1, \dots, b_N be the elements of

$$\{\sigma^j(a_i) \mid 1 \leq i \leq \ell, 0 \leq j < h\}$$

written in some order, so $N = \ell h$. Then R is also generated by b_1, \dots, b_N , and the corresponding vanishing ideal in the difference polynomial ring $\mathbb{Z}\{Y_1, \dots, Y_N\}$ is generated as a perfect

difference ideal by difference polynomials of order one. Replacing a_1, \dots, a_ℓ by b_1, \dots, b_N , we may assume that I is already generated as a perfect difference ideal by order one difference polynomials.

Let \mathfrak{p} be the ideal of all polynomials in x_1, \dots, x_ℓ vanishing on (a_1, \dots, a_ℓ) and let $\mathfrak{q} \subseteq \mathbb{Z}[x_1, \dots, x_\ell, y_1, \dots, y_\ell]$ be the ideal of all polynomials vanishing on $(a_1, \dots, a_\ell, \sigma(a_1), \dots, \sigma(a_\ell))$. If K has positive characteristic, let p be its characteristic. Otherwise, let p be any sufficiently large prime so that the ideal $(p) + \mathfrak{q}$ is proper. Let \mathfrak{Q} be a minimal prime extending $(p) + \mathfrak{q}$ and $\mathfrak{P} := \mathfrak{Q} \cap \mathbb{Z}[x_1, \dots, x_\ell]$. Then

$$X := \text{Spec}(\mathbb{Z}[x_1, \dots, x_\ell]/\mathfrak{P}) \text{ and } C := \text{Spec}(\mathbb{Z}[x_1, \dots, x_\ell, y_1, \dots, y_\ell]/\mathfrak{Q})$$

are irreducible schemes of finite type over \mathbb{F}_p . Then, by [35, Theorem 0.1], there exists a positive integer s such that the intersection of C with the graph of ϕ_s is nonempty. Let $(a_1^*, \dots, a_\ell^*, \phi_s(a_1^*), \dots, \phi_s(a_\ell^*))$ be a point in the intersection. Since the substitution $\sigma^j(x_i) = \phi_s^j(a_i^*)$ annihilates \mathfrak{q} , it also annihilates every polynomial in its perfect closure I (see (18)). Let \mathbb{F} be the algebraic closure of \mathbb{F}_p . Then the map $\psi: R \rightarrow \mathbb{F}$ defined by $\psi(\sigma^j(a_i)) = \phi_s^j(a_i^*)$ is a desired homomorphism of difference rings (R, σ) and (\mathbb{F}, ϕ_s) . \square

Lemma 8. *For every*

- *prime number p ,*
- *positive integer s ,*
- *scheme X of finite type defined over \mathbb{F} , the algebraic closure of \mathbb{F}_p ,*
- *irreducible subvariety $\Gamma \subset X \times \phi_s(X)$ such that the projections to X and $\phi_s(X)$ are dominant,*

there exist positive integers ℓ and t and an infinite sequence $(a_i)_{i=-\infty}^\infty$ such that $(a_i, a_{i+1}) \in \phi_{si}(\Gamma)$ for every $i \in \mathbb{Z}$.

Proof. Since X and Γ are defined over some finite subfield of \mathbb{F} , there is a positive integer ℓ with $\phi_{s\ell}(X) = X$ and $\phi_{s\ell}(\Gamma) = \Gamma$. Lemma 3 implies that there exists an irreducible component Ξ of the fiber product

$$\Gamma \times_{\phi_s(X)} \phi_s(\Gamma) \times_{\phi_{2s}(X)} \cdots \times_{\phi_{(\ell-1)s}(X)} \phi_{(\ell-1)s}(\Gamma).$$

such that the projections of Ξ onto $\Gamma, \phi_s(\Gamma), \dots, \phi_{(\ell-1)s}(\Gamma)$ are dominant. We denote the projection $\Xi \rightarrow \phi_{si}(\Gamma)$ by ρ_i for every $0 \leq i \leq \ell - 1$.

Let $\tau_1: \Gamma \rightarrow X$ and $\tau_2: \Gamma \rightarrow \phi_s(X)$ denote the projections. We define projections $\pi_i: \Xi \rightarrow \phi_{si}(X)$ for $0 \leq i \leq \ell$ as follows:

$$\pi_i = \begin{cases} \tau_1 \circ \rho_0, & \text{for } i = 0, \\ \tau_1 \circ \rho_i = \tau_2 \circ \rho_{i-1}, & \text{for } 0 < i < \ell, \\ \tau_2 \circ \rho_{\ell-1}, & \text{for } i = \ell. \end{cases}$$

Consider the fiber product of $\Xi \times_X \Xi$ where the first $\Xi \rightarrow X$ is π_ℓ and the second map $\Xi \rightarrow X$ is π_0 . Lemma 3 implies that there exists an irreducible component Υ of this product such that the projections of Υ onto both Ξ 's are dominant. Take r so that Ξ and Υ are both defined over \mathbb{F}_{p^r} and

$$s\ell \mid r.$$

By [35, Theorem 0.1], there is a power ϕ_t of ϕ_r and a point $a \in \Xi(\mathbb{F})$ with $(a, \phi_t(a)) \in \Upsilon(\mathbb{F})$. Note that ϕ_t leaves invariant Γ, Ξ , and Υ . Since coefficients of π_i 's and ρ_j 's are invariant under

ϕ , we will denote the conjugation of any of these maps by any power of ϕ by the same letter. For $0 \leq i < \ell$ and $j \in \mathbb{Z}$, define

$$a_{i+j\ell} := \pi_i(\phi_{t_j}(a)).$$

Let us show that the sequence $\{a_i\}_{i=-\infty}^{\infty}$ satisfies the requirement of the lemma. Consider $j \in \mathbb{Z}$ and $0 \leq i < \ell$. Then $a_{i+j\ell} = \tau_1(\rho_i(\phi_{t_j}(a)))$. We also have

$$\begin{aligned} a_{i+j\ell+1} &= \tau_1(\rho_{i+1}(\phi_{t_j}(a))) = \tau_2(\rho_i(\phi_{t_j}(a))) \text{ for } i < \ell - 1, \\ a_{i+j\ell+1} &= \tau_1(\rho_0(\phi_{t_{(j+1)}}(a))) = \tau_2(\rho_{\ell-1}(\phi_{t_j}(a))) \text{ for } i = \ell - 1 \end{aligned}$$

because Ξ and Υ are components of the corresponding fiber products. In both cases,

$$(a_{i+j\ell}, a_{i+j\ell+1}) \in \rho_i(\phi_{t_j}(\Xi)) = \rho_i(\Xi) \subset \phi_{s_i}(\Gamma) = \phi_{s(i+j\ell)}(\Gamma). \quad \square$$

In Lemmas 9 and 10, for a valued field (K, v) , we write

$$\mathcal{O} = \{x \in K : v(x) \geq 0\}$$

for the valuation ring,

$$\mathfrak{m} = \{x \in K : v(x) > 0\}$$

for the maximal ideal of \mathcal{O} , and $k = \mathcal{O}/\mathfrak{m}$ for the residue field. We denote the reduction map $r : \mathcal{O} \rightarrow k$ by r and will abuse notation writing r for the reduction map on associated objects.

Lemma 9. *Let (K, v) be a Henselian field, $n \leq m$ positive integers, $f_1, \dots, f_n \in \mathcal{O}[x_1, \dots, x_m]$ and $a = (a_1, \dots, a_m) \in k^m$. We assume that, for each i , we have $r(f_i)(a) = 0$ and that the matrix $\left(\frac{\partial r(f_i)}{\partial x_j}(a)\right)_{1 \leq i \leq n, 1 \leq j \leq m}$ has rank n . Then there is $c = (c_1, \dots, c_m) \in \mathcal{O}^m$ such that $f_1(c) = \dots = f_n(c) = 0$ and $r(c) = a$.*

Proof. By hypothesis, there is some $J \subseteq \{x_1, \dots, x_m\}$ with $|J| = n$ and invertible matrix $\left(\frac{\partial r(f_i)}{\partial x_j}(a)\right)_{1 \leq i \leq n, j \in J}$. Relabeling the variables, we may assume that $J = \{1, \dots, n\}$. Define $f_i := x_i$ for $n < i \leq m$. Then the square matrix $\left(\frac{\partial r(f_i)}{\partial x_j}(a)\right)_{1 \leq i \leq n, 1 \leq j \leq n}$ is invertible. There exists some $b \in \mathcal{O}^m$ such that $r(b) = a$. Then, by [24, Section 4, Multidimensional Hensel's Lemma], there is some $c \in \mathcal{O}^n$ with $f_1(c) = \dots = f_n(c) = 0$ and $r(c) = r(a)$. \square

Lemma 10. *Let (K, v) be a Henselian field and $f : X \rightarrow Y$ a smooth map of schemes of finite type over \mathcal{O} . Suppose that $a \in X(k)$ and $b \in Y(\mathcal{O})$ satisfy $f(a) = r(b)$. Then there is a point $c \in X(\mathcal{O})$ with $f(c) = b$ and $r(c) = a$.*

Proof. [1, Lemma 28.32.11] implies that there are affine open neighborhoods $U \subseteq X$ and $V \subseteq Y$ of a and $r(b)$, respectively, for which $f_U : U \rightarrow V$ is standard smooth. That is, there exist:

- positive integers m and n ,
- a finitely generated \mathcal{O} -algebra S ,
- polynomials $g_1, \dots, g_n \in S[x_1, \dots, x_m]$, and
- isomorphisms of schemes $\varphi_1 : U \rightarrow \text{Spec}(T)$ and $\varphi_2 : V \rightarrow \text{Spec}(S)$, where $T = S[x_1, \dots, x_m]/(g_1, \dots, g_n)$,

such that

- some $n \times n$ minor of the Jacobian $\left(\frac{\partial g_i}{\partial x_j}\right)$ is an invertible element of T and
- $f_U = \varphi_2^{-1} \circ g \circ \varphi_1$, where $g : \text{Spec}(T) \rightarrow \text{Spec}(S)$ is the dual morphism of schemes to the \mathcal{O} -algebra homomorphism $S \rightarrow T$.

Since \mathcal{O} is a local ring and $r(b)$ is a reduction of b modulo $\mathfrak{m} \subset \mathcal{O}$, the point b belongs to any open neighborhood of $r(b)$, in particular, $b \in V(\mathcal{O})$. By pre-composing with the dual to φ_2 , this corresponds to an \mathcal{O} -algebra homomorphism $b^\sharp: S \rightarrow \mathcal{O}$. Let a^\sharp denote the \mathcal{O} -algebra homomorphism obtained by pre-composing the dual to φ_1 with the \mathcal{O} -algebra homomorphism $\mathcal{O}[U] \rightarrow k$ corresponding to $a \in U(k)$.

For each i , $1 \leq i \leq n$, consider the polynomials $g_i^{b^\sharp} \in \mathcal{O}[x_1, \dots, x_m]$ and $g_i^{a^\sharp} \in k[x_1, \dots, x_m]$. The fact $r(b) = f(a)$ implies that $r(g_i^{b^\sharp}) = g_i^{a^\sharp}$. Let $a_j := a^\sharp(x_j)$ for $1 \leq j \leq m$. Since a^\sharp is a homomorphism, $g_i^{a^\sharp}(a_1, \dots, a_m) = 0$ for every $1 \leq i \leq n$ and also the Jacobian matrix $\left(\frac{\partial g_i^{a^\sharp}}{\partial x_j}\right)$ has full rank at (a_1, \dots, a_m) .

Then, by Lemma 9, we may find $(c_1, \dots, c_m) \in \mathcal{O}^m$ for which $g_i^{b^\sharp}(c_1, \dots, c_m) = 0$ for $1 \leq i \leq n$ and $r(c_j) = a_j$ for $1 \leq j \leq m$. Since $g_i^{b^\sharp}(c_1, \dots, c_m) = 0$ for $1 \leq i \leq n$, the map $c^\sharp: T \rightarrow \mathcal{O}$ defined by $c^\sharp|_S = b^\sharp$ and by $c^\sharp(x_i) = c_i$ for $1 \leq i \leq m$ is a well-defined \mathcal{O} -algebra homomorphism. This gives us a point $c \in U(\mathcal{O})$ such that $f(c) = b$. Moreover, $r(c) = a$ since $r(c_j) = a_j$ for every $1 \leq j \leq m$. \square

Corollary 6. *Let (K, v) be a Henselian field and X a scheme of finite type over \mathcal{O} such that the canonical morphism $X \rightarrow \text{Spec}(\mathcal{O})$ is smooth. Then, for every $a \in X(k)$, there exists $c \in X(\mathcal{O})$ such that $r(c) = a$.*

Proof. The corollary follows from Lemma 10 applied to $Y = \text{Spec}(\mathcal{O})$ and b being the identity map $\text{Spec}(\mathcal{O}) \rightarrow \text{Spec}(\mathcal{O})$. \square

With these statements in place, we finish the proof of Proposition 4.

In what follows, for a positive integer m and a commutative ring R , R^m denotes the commutative ring generated by the set $\{r^m \mid r \in R\}$. For an affine scheme X over a perfect ring R of characteristic p and $q = p^n$, we define a scheme $X^{(q)}$ by $X^{(q)} := \text{Spec}(\mathcal{O}_X^q)$. There is a map $F_n: X \rightarrow X^{(q)}$ that is dual to the inclusion $\mathcal{O}_X^q \hookrightarrow \mathcal{O}_X$. This map is a special case of what is called the relative Frobenius morphism. See [1, Tag 0CC6] for more details. In coordinates, when $X \subset \mathbb{A}^m$, F_n can be written as $(a_1, \dots, a_m) \mapsto (a_1^q, \dots, a_m^q)$. Since R is perfect, F_n defines a bijection between $X(R)$ and $X^{(q)}(R)$. If $p = 0$, we will assume that $q = 1$ and F_n is the identity map.

Lemma 11. *If $\mu: \Gamma \rightarrow Z$ is morphism of irreducible affine varieties over an algebraically closed field K , then there exist*

- an affine variety Υ ,
- morphisms $\nu: \Gamma \rightarrow \Upsilon$ and $\tau: \Upsilon \rightarrow Z$,
- a positive integer n and a morphism $\gamma: \Upsilon \rightarrow \Gamma^{(q)}$, where $q = p^n$,

such that $\mu = \tau \circ \nu$, $\gamma \circ \nu = F_n$, ν is finite, and τ is generically smooth.

Proof. If $\text{char } K = 0$, take $\Upsilon = \Gamma$, $\nu = \text{id}_\Gamma$ and $\tau = \mu$ by [31, Theorem 2.27].

Let $\text{char } K = p > 0$, t_1, \dots, t_ℓ be a transcendence basis of $K(\Gamma)$ over $E := \text{Quot}(\mu^*(\mathcal{O}_Z))$, and L be the relative separable closure of $E(t_1, \dots, t_\ell)$ in $K(\Gamma)$. Then, as $K(\Gamma)$ is a finite purely inseparable extension of L , for $n \gg 0$ we have $K(\Gamma)^{p^n} \subseteq L$. Let $q := p^n$ and $A = \mu^*(\mathcal{O}_Z)[\mathcal{O}_\Gamma^q]$, the ring generated by $\mu^*(\mathcal{O}_Z)$ and \mathcal{O}_Γ^q . Set $\Upsilon := \text{Spec}(A)$ over K .

Dual to the homomorphisms of rings $\mathcal{O}_Z \rightarrow A$ and $A \rightarrow \mathcal{O}_\Gamma$, we have morphisms $\tau: \Upsilon \rightarrow Z$ and $\nu: \Gamma \rightarrow \Upsilon$ with $\mu = \tau \circ \nu$. Since the field extensions $E \hookrightarrow K(\Upsilon)$ is a subextension of the the separable extension $E \hookrightarrow L$, the morphism $\tau: \Upsilon \rightarrow Z$ is smooth at the generic point of

Υ due to [1, Lemma 10.138.9]. Form the inclusion $\mathcal{O}_\Gamma^q \hookrightarrow A = \mathcal{O}_\Upsilon$, we obtain the morphism $\gamma: \Upsilon \rightarrow \Gamma^{(q)}$ with $F_n = \gamma \circ \nu$.

Since $\mathcal{O}_\Gamma^q \subset \mathcal{O}_\Gamma$ is a finite integral extension and $\mathcal{O}_\Gamma^q \subset A \subset \mathcal{O}_\Gamma$, the extension $A \subset \mathcal{O}_\Gamma$ is also a finite integral extension. Hence, the dual map $\nu: \Gamma \rightarrow \Upsilon$ is a finite morphism. \square

With these lemmas in place, we finish the proof of Proposition 4.

Proof of Proposition 4. The only implication whose proof in the original argument for Proposition 1 used uncountability is from 5. to 1. We observe that 5. implies 3., because 1 is not contained in any ideal generated by finitely many transforms of the system, so Hilbert's Nullstellensatz implies that there exist arbitrarily long partial solutions of the system over K . This is exactly 3.

Consider a triple (X, π_1, π_2) constructed in Section 6.2. Due to Lemma 1, 3. implies that (X, π_1, π_2) has an arbitrarily long partial solution. On the other hand, the existence of a solution to S in $K^{\mathbb{Z}}$ is equivalent to the existence of a two-sided infinite solution to (X, π_1, π_2) over K (see Lemma 1). We thus reduce to finding a solution to (X, π_1, π_2) over K . Then Corollary 4 implies that there exists an infinite skew-cyclic train

$$(19) \quad (\dots, Y_1, Y_2, \dots, Y_\ell, \sigma^\ell(Y_1), \dots)$$

in X . Let Z be an irreducible subvariety of the fiber product as in Lemma 5 applied to the train (Y_1, \dots, Y_ℓ) . For $1 \leq i \leq \ell$, let $\rho_i: Z \rightarrow Y_i$ be the dominant projection to Y_i (which is $\psi_{\ell,i}|_Z$ in the notation of Lemma 5). The projection $\sigma^j(Z) \rightarrow \sigma^j(Y_i)$ obtained by conjugation by σ^j of ρ_i will be denoted by $\sigma^j(\rho_i)$ for every $j \in \mathbb{Z}$. Recall that, since (19) is a train, $\pi_2 \circ \rho_\ell$ and $\pi_1 \circ \sigma^\ell(\rho_1)$ are dominant onto the same variety. Due to Lemma 3, there exists an irreducible component Γ , which we fix, of the fiber product of Z with $\sigma^\ell(Z)$ over $\pi_2 \circ \rho_\ell$ and $\pi_1 \circ \sigma^\ell(\rho_1)$ such that $\mu_1: \Gamma \rightarrow Z$ and $\mu_2: \Gamma \rightarrow \sigma^\ell(Z)$ are dominant.

Let us call a sequence $(a_i)_{i=-\infty}^\infty$ with $a_i \in \sigma^{i\ell}(Z)$ and $(a_i, a_{i+1}) \in \sigma^{i\ell}(\Gamma)$ for all i a *weak solution* to (Z, Γ) . Such a weak solution gives rise to the solution

$$(\dots, \sigma^{-\ell}(\rho_1)(a_{-1}), \dots, \sigma^{-\ell}(\rho_\ell)(a_{-1}), \rho_1(a_0), \rho_2(a_0), \dots, \rho_\ell(a_0), \sigma^\ell(\rho_1)(a_1), \dots, \sigma^\ell(\rho_\ell)(a_1), \dots)$$

of (X, π_1, π_2) . Thus, it suffices for us to find a weak solution.

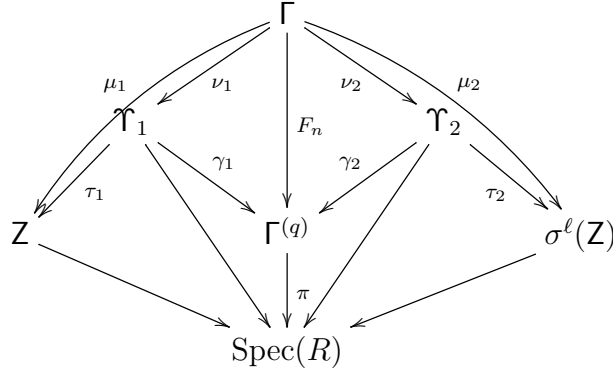
- Υ_i and Υ_2 be affine varieties
- n a positive integer,
- $\nu_i: \Gamma \rightarrow \Upsilon_i$, $\tau_i: \Upsilon_i \rightarrow Z$, $\gamma_i: \Upsilon_i \rightarrow \Gamma^{(q)}$, morphisms, where $q = p^n$ and $i = 1, 2$,

so that, for $i = 1, 2$,

- $\gamma_i \circ \nu_i = F_n$,
- τ_i is generically smooth,
- $\mu_i = \tau_i \circ \nu_i$,

which is possible to do by Lemma 11.

We fix some equations defining Γ , Z , Υ_i , τ_i , ν_i , and μ_i for $i = 1, 2$. Denote the difference ring generated by the coefficients of these equations by R . Let Γ , Z , Υ_1 , and Υ_2 be the models of Γ , Z , Υ_1 , and Υ_2 defined by these fixed equations over R . Thus, we have the following diagram:



Let $\hat{\Upsilon}_1$, $\hat{\Upsilon}_2$, and $\hat{\Gamma}^{(q)}$ be dense open subsets in Υ_1 , Υ_2 , and $\Gamma^{(q)}$, respectively, such that τ_1 , τ_2 , and π , respectively, are smooth on these subsets, which exist since smoothness of a morphism is an open condition (see the discussion just after [1, Definition 28.32.1]). Let

$$\tilde{\Gamma} = \nu_1^{-1}(\hat{\Upsilon}_1) \cap \nu_2^{-1}(\hat{\Upsilon}_2) \cap F_n^{-1}(\hat{\Gamma}^{(q)}),$$

which is dense open in Γ . Let Γ' be a non-empty open subset of $\tilde{\Gamma}$ is defined by a single inequality $f \neq 0$, where $f \in \mathcal{O}_\Gamma$. The image of Γ' under F_n is open dense in $(\Gamma')^{(q)} \subset \Gamma^{(q)}$, defined by $f^q \neq 0$. Let $\Upsilon'_i = \gamma_i^{-1}((\Gamma')^{(q)}) \cap \hat{\Upsilon}_i$, $i = 1, 2$. Then $\nu_i(\Gamma') \subset \Upsilon'_i$, and $(\Gamma')^{(q)} \subset \hat{\Gamma}^{(q)}$.

We apply Lemma 7 to (R, σ^ℓ) and obtain $\psi: (R, \sigma^\ell) \rightarrow (\mathbb{F}, \phi_s)$, where \mathbb{F} is the algebraic closure of \mathbb{F}_p and, in the case $\text{char } K = 0$, p is some prime number provided by Lemma 7. Let $X_{\mathbb{F}}$ denote the base change of a scheme X over R to \mathbb{F} via ψ . Let $(a_i)_{i=-\infty}^{\infty}$ be a sequence such that, for each $i \in \mathbb{Z}$,

$$(a_i, a_{i+1}) \in \phi_{si}(\Gamma'_{\mathbb{F}})(\mathbb{F}).$$

Such a sequence exists by Lemma 8. Fix an extension of ψ to a place ϑ on K (see [12, Theorem 3.1.1]). Let \mathcal{O} be the valuation ring of ϑ and v be a valuation on K . Note that $R \subset \mathcal{O}$. Also note that we do not assert that ϑ respects σ on all of \mathcal{O} nor even that \mathcal{O} is preserved by σ . Let \mathbb{E} be the residue field of \mathcal{O} . Since K is algebraically closed, \mathbb{E} is also algebraically closed [12, Theorem 3.2.11]. Since $\mathbb{F}_p \subset \mathbb{E}$, \mathbb{F} is embedded into \mathbb{E} .

[1, Lemma 28.32.5] implies that the morphisms of schemes $(\Upsilon'_1)_{\mathcal{O}} \rightarrow Z_{\mathcal{O}}$, $(\Upsilon'_2)_{\mathcal{O}} \rightarrow \sigma^\ell(Z)_{\mathcal{O}}$, and $(\Gamma')_{\mathcal{O}}^{(q)} \rightarrow \text{Spec}(\mathcal{O})$ are smooth as well as all their shifts/conjugations by σ^ℓ . We shall now build a weak solution $(b_i)_{i=-\infty}^{\infty}$ to $(Z_{\mathcal{O}}(\mathcal{O}), \Gamma'_{\mathcal{O}}(\mathcal{O}))$ so that

$$\vartheta(b_i) = a_i \text{ for each } i \in \mathbb{Z}.$$

Since K is algebraically closed, (K, v) is Henselian [24, Lemma 4.1]. For $i = 0$ and $i = 1$, since $(\Gamma')_{\mathcal{O}}^{(q)} \rightarrow \text{Spec}(\mathcal{O})$ is smooth, every point in $(\Gamma')_{\mathbb{F}}^{(q)}(\mathbb{F})$ lifts to a point in $(\Gamma')_{\mathcal{O}}^{(q)}(\mathcal{O})$ due to Corollary 6. Thus, we may choose some $(\hat{b}_0, \hat{b}_1) \in (\Gamma')_{\mathcal{O}}^{(q)}(\mathcal{O})$ specializing to $(F_n(a_0), F_n(a_1))$ and set $b_0 = F_n^{-1}(\hat{b}_0)$ and $b_1 = F_n^{-1}(\hat{b}_1)$.

Assume that we have already constructed b_i for some $i > 0$ so that $\vartheta(b_i) = a_i$. Due to Lemma 10 applied to the morphism of schemes $\sigma^{i\ell} \circ \tau_1 \circ \sigma^{-i\ell} : \sigma^{i\ell}((\Upsilon'_1)_{\mathcal{O}}) \rightarrow \sigma^{i\ell}(Z_{\mathcal{O}})$ and points $(\sigma^{i\ell} \circ \nu_1 \circ \sigma^{-i\ell})((a_i, a_{i+1}))$ and b_i , there exists $P \in \sigma^{i\ell}((\Upsilon'_1)_{\mathcal{O}})$ such that $(\sigma^{i\ell} \circ \tau_1 \circ \sigma^{-i\ell})(P) = b_i$ and $\vartheta(P) = (\phi_s^i \circ \nu_1 \circ \phi_s^{-i})((a_i, a_{i+1}))$. Consider

$$Q = F_n^{-1}(\sigma^{i\ell} \circ \gamma_1 \circ \sigma^{-i\ell}(P)) \in \sigma^{i\ell}(\Gamma'_{\mathcal{O}}(\mathcal{O})).$$

Since ν_1 is a finite morphism, it is surjective on \mathcal{O} -points due to [31, Theorem 1.12] together with [12, Theorem 3.1.3]. Using this and the fact that F_n is bijective on \mathcal{O} -points, $\sigma^{i\ell} \circ \nu_1 \circ$

$\sigma^{-i\ell}(Q) = P$. Hence, $(\sigma^{i\ell} \circ \mu_1 \circ \sigma^{-i\ell})(Q) = (\sigma^{i\ell} \circ \tau_1 \circ \sigma^{-i\ell})(P) = b_i$, so Q can be written as (b_i, c) . Since

$$F_n^{-1} \circ \phi_s^i \circ \gamma_1 \circ \nu_1 \circ \phi_s^{-i} = \phi_s^i \circ F_n^{-1} \circ \gamma_1 \circ \nu_1 \circ \phi_s^{-i} = \phi_s^i \circ \text{id} \circ \phi_s^{-i} = \text{id},$$

we have

$$\vartheta(Q) = F_n^{-1} \circ \phi_s^i \circ \gamma_1 \circ \nu_1 \circ \phi_s^{-i}((a_i, a_{i+1})) = (a_i, a_{i+1}).$$

Thus, we can set $b_{i+1} = c$. In the same way, we produce the b_i with $i < 0$ using the fact that $(\Upsilon'_2)_\mathcal{O} \rightarrow \sigma^\ell(\mathbb{Z})_\mathcal{O}$ is smooth. \square

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