# Definability in fields
# Lecture 3:
# Finding structure through transcendence

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# Finitely generated fields

A finitely generated field is simply a field $K$ in which there is some finite set $A = \{a_1, \ldots, a_n\} \subseteq K$ for which every element of $K$ may be written as $P(a_1, \ldots, a_n)/Q(a_1, \ldots, a_n)$ for some polynomials $P$ and $Q$ with integer coëfficients.

- Finite fields
- Number fields
- Fields of rational functions $k(t)$ over finitely generated fields
- Finite extensions of finitely generated fields
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# Algebraic varieties over $\mathbb{C}$

**Proposition**

*If $V$ is a projective algebraic variety over $\mathbb{C}$, then the set of meromorphics functions on $\mathbb{C}$ is naturally a finitely generated field over $\mathbb{C}$ and every field finitely generated over $\mathbb{C}$ has this form.*

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More algebraically, if $V \subseteq \mathbb{C}^n$ is an irreducible affine complex algebraic variety, then $I(V) := \{f \in \mathbb{C}[X_1, \ldots, X_n] \mid (\forall \mathbf{a} \in V) f(\mathbf{a}) = 0\}$ is a prime ideal and the field of rational functions on $V$, $\mathbb{C}(V)$, may be expressed as the field of fractions of $\mathbb{C}[X_1, \ldots, X_n]/I(V)$.

# Finitely generated fields as function fields

In general, if $K$ is finitely generated over the field $k$, then $K$ may be expressed as a function field over $k$.

- Choose generators $a_1, \ldots, a_n \in K$.
- The ideal $\mathfrak{p} := I(\mathbf{a}/k) := \{f \in k[X_1, \ldots, X_n] \mid f(\mathbf{a}) = 0\}$ is prime.
- If $V$ is the algebraic variety defined by $\mathfrak{p}$, then $K \cong k(V)$.

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# How much does $k(V)$ know about $V$?

- The association from an algebraic variety $V$ over the field $k$ to the finitely generated (over $k$) field $k(V)$ is well-defined.
- Our construction of an inverse requires the choice of generators.
- In general, non-isomorphic varieties may yield the same field.
- In fact, there need not even be a "best choice" of a variety.

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# Function fields of smooth curves

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**Theorem**

*The association $C \mapsto k(C)$ is an equivalence of categories between the category of smooth, projective, absolutely irreducible, curves over the field $k$ and the category of finitely generated (over $k$) fields of transcendence degree one over $k$.*

# Changing the ground field

**Question**

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## Answer

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# A construction

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If $a_1, \ldots, a_{n-1} \in K$ are algebraically independent, then $k := \mathbb{Q}(a_1, \ldots, a_{n-1})^{\mathrm{alg}} \cap K$ is a relatively algebraically closed subfield of $K$ with $\operatorname{tr.deg}_k(K) = 1$.

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Thus, there is a unique smooth, projective, absolutely irreducible curve over $k$ for which $K = k(C)$.

# Observations about the construction

- Whilst $C$ is determined from $K$ and $k$, $k$ depends on a choice.
- $k$ does not appear to have an obvious first-order definition.

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# Natural $\mathcal{L}_{\omega_1,\omega}$ definition of algebraic dependence

The elements $a_1, \ldots, a_n$ in a field $K$ are algebraically dependent if and only if

$$K \models \bigvee_{F \in \mathbb{Z}[X_1,\ldots,X_n]} (\exists x_1, \ldots, x_n)(F(\mathbf{a}) = 0 \ \& \ F(\mathbf{x}) \neq 0)$$

# Poonen's definition of algebraic dependence

**Theorem (Poonen, after Pop)**

*For each positive integer n there is a formula $\delta_n(x_1, \ldots, x_n)$ in the language of rings having n free variables for which for any finitely generated field K and n-tuple $\mathbf{a} \in K^n$ one has $K \models \delta_n(\mathbf{a})$ if and only if the tuple is algebraically dependent.*

# Basics of quadratic forms

For $K$ a field and $\mathbf{b} = (b_1, \ldots, b_d) \in (K^\times)^d$, the Pfister form associated to $\mathbf{b}$ is

$$q_{\mathbf{b}} = \sum_{I \in {}^d 2} \mathbf{b}^I x_I^2$$

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$$q_{\mathbf{b}} = \sum_{I \in {}^d 2} \mathbf{b}^I x_I^2$$

A quadratic form $q$ represents zero if there is a nontrivial solution to the equation $q(\mathbf{a}) = 0$. It is universal if for every $r \in K^\times$ there is a solution to the equation $q(\mathbf{a}) = r$.

# Pop's definition of transcendence degree

**Theorem (Pop)**

*If $K$ is a finitely generated field of characteristic zero, then* $\mathrm{tr.deg}(K) = d$ *if and only if for every* $d+2$*-tuple* $(b_1, \ldots, b_{d+2}) \in (K[\sqrt{-1}]^\times)^{d+2}$ *the form* $q_{\mathbf{b}}$ *is universal while for some choice the form does not represent zero.*

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The proof uses Voevodsky's theorem on the Milnor conjecture relating Galois cohomology groups, Milnor $K$-groups, and Witt groups.

# Pop's near definition of $\delta$

**Theorem (Pop)**

*If $K$ is a finitely generated field of characteristic zero and transcendence degree $d$, then if $c$ and $d$ are algebraic numbers for which $q_{(t_1,\ldots,t_d,c,d)}$ does not represent zero over $K[\sqrt{-1}]$, then $(t_1,\ldots,t_d)$ are algebraically independent. Almost conversely, if $(t_1,\ldots,t_d)$ is a transcendence basis, then for many choices of $(a_1,\ldots,a_d,c,d) \in \mathbb{Z}^{d+2}$, the form $q_{(t_1-a_1,\ldots,t_d-a_d,c,d)}$ does not represent zero.*

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Poonen's refinement is proven by showing that the relevant algebraic and integer points may be recognized as the coördinates of points on certain elliptic curves.

- The algebraic closure of the prime field is definable as $k := \delta_1(K)$.
- If $a_1, \ldots, a_d \in K$ are algebraically independent, then the relative algebraic closure of the field generated by $a_1, \ldots, a_d$ is (parametrically) definable as $\delta_{d+1}(K; \mathbf{a})$.
- Consequently, every infinite finitely generated field has an undecidable theory.
- If $K$ is a finitely generated field of positive transcendence degree, then there is a (parametrically) definable relatively algebraically closed subfield $k \subseteq K$ for which $\operatorname{tr.deg}_k(K) = 1$.

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# Know thyself

If $K$ is a finitely generated field of transcendence degree at least one, then we can express $K$ as $K = k(C)$ where $k$ is a (parametrically) definable relatively algebraically closed subfield and $C$ is a smooth projective curve over $k$.

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Working by induction on the transcendence degree and using Rumely's theorem in the base case, we may assume that $k$ is (parametrically) biïnterpretable with $\mathbb{Z}$. In particular, every arithmetic (relative to the standard recursive presentation of $k$) set in $k$ is definable.

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To conclude that $K$ is biïnterpretable with $\mathbb{Z}$ it would suffice for it to recognize itself as a field of functions.

# Valuations on curves

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More generally, for any smooth projective curve $C$ over $k$ and closed point $P \in C$ there is a valuation $\mathrm{ord}_P$ on $k(C)$ given by order of vanishing at $P$ and $f(P) = b$ just in case $\mathrm{ord}_P(f - b) > 0$.

# Which local-global principle?

To define the valuations on function fields of curves we use a local-global principle for Brauer groups (proven by Auslander and Brumer), though if one ignores characteristic two the Witt index theorem would suffice.

# Central simple algebras

Let $\ell$ be a prime, $K$ a field of characteristic different from $\ell$, $\omega \in K^\times$ an $\ell^{\text{th}}$ root of unity, and $A$ and $B$ two nonzero elements of $K$. Then $D(A, B, \omega; K)$ is the noncommutative $K$-algebra generated by $\alpha$ and $\beta$ subject to the relations $\alpha^\ell = A$, $\beta^\ell = B$, and $\beta\alpha = \omega\alpha\beta$.

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**Theorem**

*$D(A, B, \omega; K)$ is a division ring if and only if $A$ is not an $\ell^{th}$ power in $K$ and $B$ is not a norm from $K(\sqrt[\ell]{A})$.*

# Auslander-Brumer local-global principle

## Theorem

*Let $\ell$ be a prime, $k$ be a field, $\omega \in k$ an $\ell^{th}$ root of unity, $A, B \in k(t)^\times$ two nonzero rational functions over $k$. Then $D(A, B, \omega; k(t))$ is a division ring if and only if there is some completion $K$ of $k(t)$ with respect to a valuation trivial on $k$ for which $D(A, B, \omega; K)$ is a division ring.*

# Defining valuations

- Using the local-global principle, we cook up a couple of rings depending on the parameter $f \in k(t)$ which will be division rings just in case $\mathrm{ord}_a(f) \equiv 0 \pmod{\ell}$.
- To be honest, the formulas in question only work when $k$ is replaced by some kind of complete field and a density argument is required to encode everything in $k(t)$.
- One extends to $k(C)$ for more general curves analogously to J. Robinson's method of studying number rings.

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# Geometric problems

- Is $\mathrm{Th}(\mathbb{C}(t))$ decidable?
- Is $\mathbb{C}[t]_{(t)}$ parametrically definable in $\mathbb{C}(t)$?
- If $K$ and $L$ are finitely generated over the same algebraically closed field $k$ and $K \equiv L$, must we have $K \cong L$?

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# Arithmetic problems

- Is there a straightforward way to deduce the existence of Gödel coding in finitely generated fields from the Pop/Poonen transcendence definition?
- Is there an alternate way to demonstrate undecidabilty of the theory of $\mathbb{Q}$ without directly interpreting $\mathbb{Z}$?
- Is there a uniform biïnterpretation of infinite finitely generated fields with $\mathbb{Z}$?
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- $\mathbb{Q}^{\mathrm{alg}}(t, s)$ and $\mathbb{Z}$ interpret each other. Are they biïnterpretable?