Definability in fields
Lecture 1:
Undecidabile arithmetic, decidable geometry

Thomas Scanlon

University of California, Berkeley

5 February 2007
Model Theory and Computable Model Theory
Gainesville, Florida

Structures from logic

Question

What do we study when we examine mathematical structures from the perspective of logic?

- What formal sentences are true in $\mathfrak{M}$?
- What sets are definable in $\mathfrak{M}$?

Structures from logic

Question

What do we study when we examine mathematical structures from the perspective of logic?

Given an $\mathscr{L}$-structure $\mathfrak{M}$ we might ask:

- What formal sentences are true in $\mathfrak{M}$?
- What sets are definable in $\mathfrak{M}$?

Structures from logic

Question

What do we study when we examine mathematical structures from the perspective of logic?

Given an $\mathscr{L}$-structure $\mathfrak{M}$ we might ask:

- What formal sentences are true in $\mathfrak{M}$?
- What sets are definable in $\mathfrak{M}$?

Structures from logic

Question

What do we study when we examine mathematical structures from the perspective of logic?

Given an $\mathscr{L}$-structure $\mathfrak{M}$ we might ask:

- What formal sentences are true in $\mathfrak{M}$? That is, what is $\mathrm{Th}_{\mathscr{L}}(\mathfrak{M}) := \{\varphi \mid \mathfrak{M} \models \varphi\}$.
- What sets are definable in $\mathfrak{M}$?

Structures from logic

Question

What do we study when we examine mathematical structures from the perspective of logic?

Given an $\mathscr{L}$-structure $\mathfrak{M}$ we might ask:

- What formal sentences are true in $\mathfrak{M}$? That is, what is $\mathrm{Th}_{\mathscr{L}}(\mathfrak{M}) := \{\varphi \mid \mathfrak{M} \models \varphi\}$. Perhaps more importantly, how do we decide which sentences are true in $\mathfrak{M}$?
- What sets are definable in $\mathfrak{M}$?

Structures from logic

Question

What do we study when we examine mathematical structures from the perspective of logic?

Given an $\mathscr{L}$-structure $\mathfrak{M}$ we might ask:

- What formal sentences are true in $\mathfrak{M}$? That is, what is $\mathrm{Th}_{\mathscr{L}}(\mathfrak{M}) := \{\varphi \mid \mathfrak{M} \models \varphi\}$. Perhaps more importantly, how do we decide which sentences are true in $\mathfrak{M}$?
- What sets are definable in $\mathfrak{M}$?

Structures from logic

Question

What do we study when we examine mathematical structures from the perspective of logic?

Given an $\mathscr{L}$-structure $\mathfrak{M}$ we might ask:

- What formal sentences are true in $\mathfrak{M}$? That is, what is $\mathrm{Th}_{\mathscr{L}}(\mathfrak{M}) := \{\varphi \mid \mathfrak{M} \models \varphi\}$. Perhaps more importantly, how do we decide which sentences are true in $\mathfrak{M}$?
- What sets are definable in $\mathfrak{M}$? That is, describe the set $\mathrm{Def}(\mathfrak{M}) := \bigcup_{n=0}^{\infty} \mathrm{Def}_n(\mathfrak{M})$ where $\mathrm{Def}_n(\mathfrak{M}) := \{\varphi(\mathfrak{M}) \mid \varphi(x_1, \ldots, x_n) \in \mathscr{L}\}$ and $\varphi(\mathfrak{M}) := \{\mathbf{a} \in M^n \mid \mathfrak{M} \models \varphi(\mathbf{a})\}$.

Which question should we ask?

- Traditionally, logicians focus on decidability of theories.
- From the standpoint of logic, we can only discern a difference between structures if they satisfy different sentences. That is, elementary equivalence, $\mathfrak{M} \equiv \mathfrak{N} \Leftrightarrow \mathrm{Th}_{\mathscr{L}}(\mathfrak{M}) = \mathrm{Th}_{\mathscr{L}}(\mathfrak{N})$, is the right logical notion of two structures being the same.
- The complexity of the theory of a structure is expressed by the complexity of $\mathrm{Def}(\mathfrak{M})$.

Which question should we ask?

- Traditionally, logicians focus on decidability of theories.
- From the standpoint of logic, we can only discern a difference between structures if they satisfy different sentences. That is, elementary equivalence, $\mathfrak{M} \equiv \mathfrak{N} \Leftrightarrow \mathrm{Th}_{\mathscr{L}}(\mathfrak{M}) = \mathrm{Th}_{\mathscr{L}}(\mathfrak{N})$, is the right logical notion of two structures being the same.
- The complexity of the theory of a structure is expressed by the complexity of $\mathrm{Def}(\mathfrak{M})$.

Which question should we ask?

- Traditionally, logicians focus on decidability of theories.
- From the standpoint of logic, we can only discern a difference between structures if they satisfy different sentences. That is, elementary equivalence, $\mathfrak{M} \equiv \mathfrak{N} \Leftrightarrow \mathrm{Th}_{\mathscr{L}}(\mathfrak{M}) = \mathrm{Th}_{\mathscr{L}}(\mathfrak{N})$, is the right logical notion of two structures being the same.
- The complexity of the theory of a structure is expressed by the complexity of $\mathrm{Def}(\mathfrak{M})$.

Which question should we ask?

- Traditionally, logicians focus on decidability of theories.
- From the standpoint of logic, we can only discern a difference between structures if they satisfy different sentences. That is, elementary equivalence, $\mathfrak{M} \equiv \mathfrak{N} \Leftrightarrow \mathrm{Th}_{\mathscr{L}}(\mathfrak{M}) = \mathrm{Th}_{\mathscr{L}}(\mathfrak{N})$, is the right logical notion of two structures being the same.
- The complexity of the theory of a structure is expressed by the complexity of $\mathrm{Def}(\mathfrak{M})$.

Of course, to answer either of the questions we need to answer the other.

Specializing to rings

We focus mostly on the case of $\mathfrak{M} = (R, +, -, \times, 0, 1)$ where R is a commutative ring or even a field and we address the questions:

- Does $R \equiv S$ imply $R \cong S$ (for R and S from some fixed class of rings)? (Pop's Problem)
- Is $\mathrm{Th}(R)$ decidable?
- Is $\mathrm{Th}_\exists(R)$ decidable? (Hilbert's Tenth Problem for R)
- What is definable in $(R, +, \times)$?

Specializing to rings

We focus mostly on the case of $\mathfrak{M} = (R, +, -, \times, 0, 1)$ where R is a commutative ring or even a field and we address the questions:

- Does $R \equiv S$ imply $R \cong S$ (for R and S from some fixed class of rings)? (Pop's Problem)
- Is $\mathrm{Th}(R)$ decidable?
- Is $\mathrm{Th}_{\exists}(R)$ decidable? (Hilbert's Tenth Problem for R)
- What is definable in $(R, +, \times)$?

Specializing to rings

We focus mostly on the case of $\mathfrak{M} = (R, +, -, \times, 0, 1)$ where R is a commutative ring or even a field and we address the questions:

- Does $R \equiv S$ imply $R \cong S$ (for R and S from some fixed class of rings)? (Pop's Problem)
- Is $\text{Th}(R)$ decidable?
- Is $\text{Th}_{\exists}(R)$ decidable? (Hilbert's Tenth Problem for R)
- What is definable in $(R, +, \times)$?

Specializing to rings

We focus mostly on the case of $\mathfrak{M} = (R, +, -, \times, 0, 1)$ where R is a commutative ring or even a field and we address the questions:

- Does $R \equiv S$ imply $R \cong S$ (for R and S from some fixed class of rings)? (Pop's Problem)
- Is $\mathrm{Th}(R)$ decidable?
- Is $\mathrm{Th}_{\exists}(R)$ decidable? (Hilbert's Tenth Problem for R)
- What is definable in $(R, +, \times)$?

Specializing to rings

We focus mostly on the case of $\mathfrak{M} = (R, +, -, \times, 0, 1)$ where R is a commutative ring or even a field and we address the questions:

- Does $R \equiv S$ imply $R \cong S$ (for R and S from some fixed class of rings)? (Pop's Problem)
- Is $\mathrm{Th}(R)$ decidable?
- Is $\mathrm{Th}_\exists(R)$ decidable? (Hilbert's Tenth Problem for R)
- What is definable in $(R, +, \times)$?

Pop's problem

Conjecture

If K and L are two finitely generated fields, then $K \equiv L \Leftrightarrow K \cong L$.

Pop's problem

Conjecture

If K and L are two finitely generated fields, then $K \equiv L \Leftrightarrow K \cong L$.

In its geometric form, Pop's conjecture asserts that if K and L are finitely generated over $\mathbb{C}$, then $L \equiv K \iff L \cong K$.

An easy "solution"

If the field K had access to its own presentation, then it could describe itself.

An easy "solution"

If the field K had access to its own presentation, then it could describe itself.

A finitely generated field may be expressed as the field of quotients of a ring of the form $\mathbb{Z}[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$ where each f_i is a polynomial in n variables with integer coëfficients and $(f_1, \ldots, f_m)$ is a prime ideal.

An easy "solution"

If the field K had access to its own presentation, then it could describe itself.

A finitely generated field may be expressed as the field of quotients of a ring of the form $\mathbb{Z}[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$ where each f_i is a polynomial in n variables with integer coëfficients and $(f_1, \ldots, f_m)$ is a prime ideal.

K satisfies the first-order sentence $\exists \mathbf{a} \bigwedge f_i(\mathbf{a}) = 0$.

An easy "solution"

If the field K had access to its own presentation, then it could describe itself.

A finitely generated field may be expressed as the field of quotients of a ring of the form $\mathbb{Z}[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$ where each f_i is a polynomial in n variables with integer coëfficients and $(f_1, \ldots, f_m)$ is a prime ideal.

K satisfies the first-order sentence $\exists \mathbf{a} \bigwedge f_i(\mathbf{a}) = 0$.

K is determined up to isomorphism by the $\mathscr{L}_{\omega_1,\omega}$ sentence expressing that there is a generic solution $\mathbf{a}$ to $\bigwedge f_i(\mathbf{a})$ and every element of K is expressible as a rational functionof $\mathbf{a}$.

A very easy case of Pop's conjecture

Problem

Distinguish between $\mathbb{Q}$ *and* $\mathbb{Q}(\sqrt{2})$.

A very easy case of Pop's conjecture

Problem

Distinguish between $\mathbb{Q}$ *and* $\mathbb{Q}(\sqrt{2})$.

$$\mathbb{Q}(\sqrt{2}) \models (\exists x) x \cdot x = 1 + 1$$
$$\mathbb{Q} \models (\forall x) x \cdot x \neq 1 + 1$$

Another case of Pop's conjecture

Problem

Distinguish between $\mathbb{Q}$ *and* $\mathbb{Q}(t)$.

Another case of Pop's conjecture

Problem

Distinguish between $\mathbb{Q}$ and $\mathbb{Q}(t)$.

$$\mathbb{Q} \models (\forall x)(\exists y_1)(\exists y_2)(\exists y_3)(\exists y_4) x = y_1^2 + y_2^2 + y_3^2 + y_4^2 \vee -x = y_1^2 + y_2^2 + y_3^2 + y_4^2$$

Another case of Pop's conjecture

Problem

Distinguish between $\mathbb{Q}$ and $\mathbb{Q}(t)$.

$$\begin{aligned} \mathbb{Q} \models \; & (\forall x)(\exists y_1)(\exists y_2)(\exists y_3)(\exists y_4) x = y_1^2 + y_2^2 + y_3^2 + y_4^2 \\ & \vee \; -x = y_1^2 + y_2^2 + y_3^2 + y_4^2 \end{aligned}$$

Neither t nor $-t$ is a sum of squares in $\mathbb{Q}(t)$.

Sabbagh's question

Question (Sabbagh)

Is there a sentence τ in the language of rings for which if K is a finitely generated field of transcendence degree one, then $K \models \tau$ and if L is a finitely generated field of transcendence degree two, then $K \models \neg\tau$?

Hilbert's Tenth Problem

Problem

10. Entscheidung der Lösbarkeit einer diophantischen Gleichung. Eine diophantische Gleichung mit irgendwelchen Unbekannten und mit ganzen rationalen Zahlkoefficienten sei vorgelegt: man soll ein Verfahren angeben, nach welchen sich mittels einer endlichen Anzahl von Operationen entscheiden läßt, ob die Gleichung in ganzen rationalen Zahlen lösbar ist.

Hilbert's Tenth Problem

Problem

10. Entscheidung der Lösbarkeit einer diophantischen Gleichung. Eine diophantische Gleichung mit irgendwelchen Unbekannten und mit ganzen rationalen Zahlkoefficienten sei vorgelegt: man soll ein Verfahren angeben, nach welchen sich mittels einer endlichen Anzahl von Operationen entscheiden läßt, ob die Gleichung in ganzen rationalen Zahlen lösbar ist.

That is, find a finitistic procedure which when given a polynomial $f(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ in finitely many indeterminates over the integers determines (correctly) where or not there is a tuple $\mathbf{a} \in \mathbb{Z}^n$ with $f(\mathbf{a}) = 0$.

Matiyasevich's theorem (first form)

Theorem (Matiyasevich (using Davis-Putnam-(J.) Robinson))

There is no solution to Hilbert's Tenth Problem.

Gödel's Incompleteness Theorems

Theorem (First Incompleteness Theorem)

$\mathrm{Th}(\mathbb{Z}, +, \times)$ *is undecidable.*

Gödel's Incompleteness Theorems

Theorem (First Incompleteness Theorem)

$\mathrm{Th}(\mathbb{Z}, +, \times)$ *is undecidable.*

Gödel actually shows that there is no decision procedure for Π^0_1-sentences. The work in the prood of the MDPR theorem involves showing that the bounded quantifiers may be encoded with Diophantine predicates.

Undecidability of $\mathbb{Q}$

Theorem (J. Robinson)

$\mathrm{Th}(\mathbb{Q}, +, \times)$ *is undecidable.*

- There is a formula $\zeta(x)$ in one free variable for which $\mathbb{Q} \models \zeta(a)$ if and only if $a \in \mathbb{Z}$. [We will discuss the construction of ζ in Lecture 2.]
- If we had a decision procedure for $\mathbb{Q}$, then we would have one for $\mathbb{Z}$ by relativizing the sentences for $\mathbb{Z}$ to $\mathbb{Q}$ using ζ.

Undecidability of $\mathbb{Q}$

Theorem (J. Robinson)

$\text{Th}(\mathbb{Q}, +, \times)$ *is undecidable.*

Proof.

- There is a formula $\zeta(x)$ in one free variable for which $\mathbb{Q} \models \zeta(a)$ if and only if $a \in \mathbb{Z}$. [We will discuss the construction of ζ in Lecture 2.]
- If we had a decision procedure for $\mathbb{Q}$, then we would have one for $\mathbb{Z}$ by relativizing the sentences for $\mathbb{Z}$ to $\mathbb{Q}$ using ζ.

□

Undecidability of $\mathbb{Q}$

Theorem (J. Robinson)

$\mathrm{Th}(\mathbb{Q}, +, \times)$ *is undecidable.*

Proof.

- There is a formula $\zeta(x)$ in one free variable for which $\mathbb{Q} \models \zeta(a)$ if and only if $a \in \mathbb{Z}$. [We will discuss the construction of ζ in Lecture 2.]
- If we had a decision procedure for $\mathbb{Q}$, then we would have one for $\mathbb{Z}$ by relativizing the sentences for $\mathbb{Z}$ to $\mathbb{Q}$ using ζ.

□

Undecidability of $\mathbb{Q}$

Theorem (J. Robinson)

$\mathrm{Th}(\mathbb{Q}, +, \times)$ *is undecidable.*

Proof.

- There is a formula $\zeta(x)$ in one free variable for which $\mathbb{Q} \models \zeta(a)$ if and only if $a \in \mathbb{Z}$. [We will discuss the construction of ζ in Lecture 2.]
- If we had a decision procedure for $\mathbb{Q}$, then we would have one for $\mathbb{Z}$ by relativizing the sentences for $\mathbb{Z}$ to $\mathbb{Q}$ using ζ.

□

Undecidability of $\mathbb{Q}$

Theorem (J. Robinson)

$\mathrm{Th}(\mathbb{Q}, +, \times)$ *is undecidable.*

Proof.

- There is a formula $\zeta(x)$ in one free variable for which $\mathbb{Q} \models \zeta(a)$ if and only if $a \in \mathbb{Z}$. [We will discuss the construction of ζ in Lecture 2.]
- If we had a decision procedure for $\mathbb{Q}$, then we would have one for $\mathbb{Z}$ by relativizing the sentences for $\mathbb{Z}$ to $\mathbb{Q}$ using ζ.

□

Hilbert's Tenth Problem for $\mathbb{Q}$ is still open. Robinson's ζ uses three alternations of quantifiers and to date no existential definition of $\mathbb{Z}$ has been found.

Undecidability of $\mathbb{F}_p(t)$

Theorem (R. Robinson)

$\mathrm{Th}(\mathbb{F}_p(t), +, \times)$ *is undecidable.*

Undecidability of $\mathbb{F}_p(t)$

Theorem (R. Robinson)

$\mathrm{Th}(\mathbb{F}_p(t), +, \times)$ *is undecidable.*

In this case, using t as a parameter, the set of powers of t is definable and Robinson shows that the set $\{(t^m, t^n, t^{mn}) : m, n \in \mathbb{Z}\}$ is also definable. Relativizing, a decision procedure for $\mathbb{F}_p(t)$ would give one for $\mathbb{Z}$.

Undecidability of $\mathbb{F}_p(t)$

Theorem (R. Robinson)

$\mathrm{Th}(\mathbb{F}_p(t), +, \times)$ *is undecidable.*

In this case, using t as a parameter, the set of powers of t is definable and Robinson shows that the set $\{(t^m, t^n, t^{mn}) : m, n \in \mathbb{Z}\}$ is also definable. Relativizing, a decision procedure for $\mathbb{F}_p(t)$ would give one for $\mathbb{Z}$.

Th. Pheidas has shown that the interpretation of $\mathbb{Z}$ may be taken to be Diophantine. Thus, Hilbert's Tenth Problem for $\mathbb{F}_p(t)$ has no solution.

Elementary geometry

Theorem (Tarski)

Elementary geometry is decidable. That is, $\mathrm{Th}(\mathbb{R})$ *is decidable.*

Elementary geometry

Theorem (Tarski)

Elementary geometry is decidable. That is, $\mathrm{Th}(\mathbb{R})$ *is decidable.*

As $\mathbb{C}$ is interpretable in $\mathbb{R}$, it follows that $\mathrm{Th}(\mathbb{C})$ is also decidable.

Elementary geometry

Theorem (Tarski)

Elementary geometry is decidable. That is, $\mathrm{Th}(\mathbb{R})$ *is decidable.*

As $\mathbb{C}$ is interpretable in $\mathbb{R}$, it follows that $\mathrm{Th}(\mathbb{C})$ is also decidable. Of course, one can deduce this as well from the theorem that the recursively axiomatized theory of algebraically closed fields of a fixed characteristic is complete.

p-adic fields

Theorem (Ax and Kochen; Eršov)

The theory of the p-adic numbers is decidable.

Valuations: Definition

Definition

A valuation v on a field K is a function $v : K \to \Gamma \cup \{\infty\}$ where $(\Gamma, +, 0, <)$ is an ordered abelian group for which for all x and y in K

- $v(x) = \infty \iff x = 0$
- $v(xy) = v(x) + v(y)$ and
- $v(x + y) \geq \min\{v(x), v(y)\}$

Valuations: Examples

Example

- K any field, $v \restriction K^\times \equiv 0$, the trivial valuation
- $K = \mathbb{Q}$, p a prime number, any $x \in \mathbb{Q}^\times$ may be expressed as $x = p^r \frac{a}{b}$ where a, b, and r are integers with a and b not divisible by p. The p-adic valuation of x is $v_p(x) := r$.
- $K = k(t)$ where k is any field and for any rational function f expressed as $f = g/h$ with g and h polynomials we set $v_\infty(f) = \deg(h) - \deg(g)$.
- If (K, v) is a valued field, then the completion $(\widehat{K}, \widehat{v})$ is also a valued field.

Valuations: Examples

Example

- K any field, $v \upharpoonright K^{\times} \equiv 0$, the trivial valuation
- $K = \mathbb{Q}$, p a prime number, any $x \in \mathbb{Q}^{\times}$ may be expressed as $x = p^r \frac{a}{b}$ where a, b, and r are integers with a and b not divisible by p. The p-adic valuation of x is $v_p(x) := r$.
- $K = k(t)$ where k is any field and for any rational function f expressed as $f = g/h$ with g and h polynomials we set $v_\infty(f) = \deg(h) - \deg(g)$.
- If (K, v) is a valued field, then the completion $(\widehat{K}, \widehat{v})$ is also a valued field.

Valuations: Examples

Example

- K any field, $v \upharpoonright K^\times \equiv 0$, the trivial valuation
- $K = \mathbb{Q}$, p a prime number, any $x \in \mathbb{Q}^\times$ may be expressed as $x = p^r \frac{a}{b}$ where a, b, and r are integers with a and b not divisible by p. The p-adic valuation of x is $v_p(x) := r$.
- $K = k(t)$ where k is any field and for any rational function f expressed as $f = g/h$ with g and h polynomials we set $v_\infty(f) = \deg(h) - \deg(g)$.
- If (K, v) is a valued field, then the completion $(\widehat{K}, \widehat{v})$ is also a valued field.

Valuations: Examples

Example

- K any field, $v \restriction K^{\times} \equiv 0$, the trivial valuation
- $K = \mathbb{Q}$, p a prime number, any $x \in \mathbb{Q}^{\times}$ may be expressed as $x = p^r \frac{a}{b}$ where a, b, and r are integers with a and b not divisible by p. The p-adic valuation of x is $v_p(x) := r$.
- $K = k(t)$ where k is any field and for any rational function f expressed as $f = g/h$ with g and h polynomials we set $v_{\infty}(f) = \deg(h) - \deg(g)$.
- If (K, v) is a valued field, then the completion $(\widehat{K}, \widehat{v})$ is also a valued field.

Valuations: Examples

Example

- K any field, $v \upharpoonright K^\times \equiv 0$, the trivial valuation
- $K = \mathbb{Q}$, p a prime number, any $x \in \mathbb{Q}^\times$ may be expressed as $x = p^r \frac{a}{b}$ where a, b, and r are integers with a and b not divisible by p. The p-adic valuation of x is $v_p(x) := r$.
- $K = k(t)$ where k is any field and for any rational function f expressed as $f = g/h$ with g and h polynomials we set $v_\infty(f) = \deg(h) - \deg(g)$.
- If (K, v) is a valued field, then the completion $(\widehat{K}, \widehat{v})$ is also a valued field. The completion of $\mathbb{Q}$ with respect to the p-adic valuation is $\mathbb{Q}_p$, the field of p-adic numbers.

Gödel's Incompleteness, revisited

The negative content of Gödel's theorem is very strong, say in the form of the Second Incompleteness theorem that if T is a consistent, recursively enumerable extension of Peano Arithmetic, then $T \nvdash \mathrm{Con}(T)$, but for us the positive content is just as striking.

Gödel's Incompleteness, revisited

The negative content of Gödel's theorem is very strong, say in the form of the Second Incompleteness theorem that if T is a consistent, recursively enumerable extension of Peano Arithmetic, then $T \nvdash \mathrm{Con}(T)$, but for us the positive content is just as striking.

Theorem (Gödel)

$\mathbb{Z}$ codes sequences in the sense that there is a formula $\sigma(x, y, z)$ in the language of rings for which

- *for any sequence $\sigma \in {}^{<\omega}\mathbb{Z}$ there is some $s \in \mathbb{Z}$ such that for any $i \in \mathbb{Z}_+$ we have $\mathbb{Z} \models \sigma(s, i, z)$ if and only if $z = \sigma(i)$,*
- $\mathbb{Z} \models (\forall s)(\forall i \geq 0)(\exists! z)\sigma(s, i, z)$

Gödel's Incompleteness, revisited

The negative content of Gödel's theorem is very strong, say in the form of the Second Incompleteness theorem that if T is a consistent, recursively enumerable extension of Peano Arithmetic, then $T \nvdash \mathrm{Con}(T)$, but for us the positive content is just as striking.

Theorem (Gödel)

$\mathbb{Z}$ codes sequences in the sense that there is a formula $\sigma(x, y, z)$ in the language of rings for which

- *for any sequence $\sigma \in {}^{<\omega}\mathbb{Z}$ there is some $s \in \mathbb{Z}$ such that for any $i \in \mathbb{Z}_+$ we have $\mathbb{Z} \models \sigma(s, i, z)$ if and only if $z = \sigma(i)$,*
- $\mathbb{Z} \models (\forall s)(\forall i \geq 0)(\exists! z)\sigma(s, i, z)$

It follows from the theorem on coding of sequences that every recursive, and more generally, every arithmetic set, is definable in $\mathbb{Z}$. Every conceivable set is definable in $(\mathbb{Z}, +, \times)$.

Definable sets in $\mathbb{Q}$

From J. Robinson's theorem on the definability of $\mathbb{Z}$ in $\mathbb{Q}$ and the usual construction of $\mathbb{Q}$ as the field of fractions of $\mathbb{Z}$, one sees that $\mathbb{Q}$ and $\mathbb{Z}$ are biïnterpretable.

Definable sets in $\mathbb{Q}$

From J. Robinson's theorem on the definability of $\mathbb{Z}$ in $\mathbb{Q}$ and the usual construction of $\mathbb{Q}$ as the field of fractions of $\mathbb{Z}$, one sees that $\mathbb{Q}$ and $\mathbb{Z}$ are biïnterpretable. Thus, every arithmetic subset of $\mathbb{Q}^n$ is definable in $(\mathbb{Q}, +, \times)$.

Definable sets in $\mathbb{Q}$

From J. Robinson's theorem on the definability of $\mathbb{Z}$ in $\mathbb{Q}$ and the usual construction of $\mathbb{Q}$ as the field of fractions of $\mathbb{Z}$, one sees that $\mathbb{Q}$ and $\mathbb{Z}$ are biïnterpretable. Thus, every arithmetic subset of $\mathbb{Q}^n$ is definable in $(\mathbb{Q}, +, \times)$.

With more work, it is possible to deduce the same result (at least as long as one is willing to use parameters in the definitions) for $\mathbb{F}_p(t)$ from R. Robinson's theorem.

Definable sets in $\mathbb{R}$

Tarski's proof of the decidability of the theory of the real numbers yields a quantifier elimination theorem.

Definable sets in $\mathbb{R}$

Tarski's proof of the decidability of the theory of the real numbers yields a quantifier elimination theorem.

Theorem (Tarski)

The real numbers admit quantifier elimination in the language of ordered rings.

Definable sets in $\mathbb{R}$

Tarski's proof of the decidability of the theory of the real numbers yields a quantifier elimination theorem.

Theorem (Tarski)

The real numbers admit quantifier elimination in the language of ordered rings.

Corollary

Every $\mathscr{L}(+, \times, 0, 1)_{\mathbb{R}}$-definable subset of $\mathbb{R}$ is a finite union of points and intervals.

Definable sets in other complete fields

Theorem (Tarski)

Algebraically closed fields eliminate quantifiers in the language of rings. Hence, every definable subset of an algebraically closed field is finite or cofinte.

Definable sets in other complete fields

Theorem (Tarski)

Algebraically closed fields eliminate quantifiers in the language of rings. Hence, every definable subset of an algebraically closed field is finite or cofinte.

Theorem

The field $\mathbb{Q}_p$ eliminates quantifiers in the language of valued fields augmented by divisibility predicates on the value group. Hence, every infinite definable subset of $\mathbb{Q}_p$ contains an open subset.

Preview

- $\mathbb{Z} = \{x \in \mathbb{Q} : (\forall v \text{ a valuation}) v(x) \geq 0\}$. We shall find uniform definitions for the valuations on $\mathbb{Q}$ by using local-global principles to relate the valuations. The decidability of each $\mathbb{Q}_p$ is essential to this project.
- Voevodsky's theorems on quadratic forms will be used to express algebraic independence.
- We will use Gödel coding in $\mathbb{Z}$ together with other local-global principles to recognize finitely generated fields as function fields.

Preview

- $\mathbb{Z} = \{x \in \mathbb{Q} : (\forall v \text{ a valuation}) v(x) \geq 0\}$. We shall find uniform definitions for the valuations on $\mathbb{Q}$ by using local-global principles to relate the valuations. The decidability of each $\mathbb{Q}_p$ is essential to this project.
- Voevodsky's theorems on quadratic forms will be used to express algebraic independence.
- We will use Gödel coding in $\mathbb{Z}$ together with other local-global principles to recognize finitely generated fields as function fields.

Preview

- $\mathbb{Z} = \{x \in \mathbb{Q} : (\forall v \text{ a valuation}) v(x) \geq 0\}$. We shall find uniform definitions for the valuations on $\mathbb{Q}$ by using local-global principles to relate the valuations. The decidability of each $\mathbb{Q}_p$ is essential to this project.
- Voevodsky's theorems on quadratic forms will be used to express algebraic independence.
- We will use Gödel coding in $\mathbb{Z}$ together with other local-global principles to recognize finitely generated fields as function fields.