DIFFERENTIAL CHOW VARIETIES EXIST

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WITH AN APPENDIX BY WILLIAM JOHNSON

Abstract. The Chow variety is a parameter space for effective algebraic cycles on $\mathbb{P}^n$ (or $\mathbb{A}^n$) of given dimension and degree. We construct its analog for differential algebraic cycles on $\mathbb{A}^n$, answering a question of [12]. The proof uses the construction of classical algebro-geometric Chow varieties, the theory of characteristic sets of differential varieties and algebraic varieties, the theory of prolongation spaces, and the theory of differential Chow forms. In the course of the proof several definability results from the theory of algebraically closed fields are required. Elementary proofs of these results are given in the appendix.

Keywords: Differential Chow variety, Differential Chow form, Prolongation admissibility, Model theory, Chow variety

1. Introduction

For simplicity in the following discussion, let k be an algebraically closed field and $\mathbb{P}^n$ the projective space over k. The r-cycles on $\mathbb{P}^n$ are elements of the free $\mathbb{Z}$-module generated by irreducible varieties in $\mathbb{P}^n$ of dimension r. If the coefficients are taken over $\mathbb{N}$ then the cycle is said to be positive or effective. For a given effective cycle $\sum n_i V_i$, the degree of $\sum n_i V_i$ is given by $\sum n_i \cdot \deg(V_i)$ where the degree of the variety is computed with respect to some fixed very ample line bundle on V. The positive r-cycles of degree d on $\mathbb{P}^n$ are parameterized by a k-variety, called the Chow variety. For background on Chow varieties and Chow forms, see [3] (or [14] or [4] for a modern exposition). The purpose of this article is to carry out the construction of the differential algebraic analog of the Chow variety, whose construction was begun in [12], but was completed only in certain very special cases. For our purposes, one can view Chow varieties and their differential counterparts as parameter spaces for cycles with particular characteristics (degree and codimension in the algebraic case). The algebraic theory of Chow varieties also has numerous applications and deeper uses (e.g. Lawson (co)homology [10] and various counting problems in geometry [6]).

Working over a differentially closed field K with a single derivation and natural number n, the group of differential cycles of dimension d and order h in affine n-space over K is the free $\mathbb{Z}$-module generated by irreducible differential subvarieties $W \subseteq \mathbb{A}^n$ so that the dimension, $\dim(W)$, is d and the order, $\mathrm{ord}(W)$, is h. The differential cycles of index (d, h, g, m) are those effective cycles with leading differential degree g and differential degree m. These invariants have a very natural

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definition and are a suitable notion of degree for differential cycles; see section 3 for the definitions. Our main result establishes the existence of a differential variety which parameterizes this particular set of effective differential cycles of $\mathbb{A}^n$.

There are various foundational approaches to differential algebraic geometry (e.g. the scheme-theoretic approaches [26] or the Weil-style definition of an abstract differential algebraic variety [24]). Such abstract settings will not be pertinent here, since we work exclusively with differential algebraic subvarieties of affine spaces over a differential field. In this setting, beyond the basic development of the theory, there are two approaches relevant to our work. The first is the classical theory using characteristic sets [23]. We also use the more recent geometric approach using the theory of jet and prolongation spaces [32]. This approach allows one to replace a differential algebraic variety by an associated sequence of algebraic varieties, but owing to the Noetherianity of the Kolchin topology, some finite portion of the sequence contains all of the data of the sequence. This allows the importation of various results and techniques from the algebraic category.

We use the two approaches in the following manner. We use classical algebraic Chow varieties to parameterize prolongation sequences. We then use the dominant components of these prolongation sequences to parameterize the *characteristic sets* of differential algebraic cycles with given index. There are essentially two steps to the construction. First, degree bounds are used to restrict the space of Chow varieties in which we must look for the points which generate the prolongation sequences parameterizing differential cycles of a given index; this development uses basic intersection theory (e.g. [16]) and the theory of differential Chow forms [12]. Within the appropriate Chow varieties which parameterize these prolongation spaces, only a subset of the points will correspond to differential cycles with the specified index. We show that this collection of points (such that the the dominant component of the differential variety corresponding to the prolongation sequence generated by each irreducible component of the algebraic cycle represented by this point has the specified index, to be defined in Definition 3.10) is in fact a differentially constructible subset.

To say more precisely what we mean by the existence of a differential Chow variety we should speak about representable functors. For a fixed ambient dimension n and index (d, h, g, m) we may associate to each differential field k the set of differential cycles of index (d, h, g, m) in affine n-space over k thereby obtaining a functor $\mathbb{C}(n, d, h, g, m) : \delta\text{- Field} \to \text{Set}$ from the category of differential fields to the category of sets. (On morphisms of differential fields, this functor is given by base change.) Of course, each differentially constructible set X (defined over $\mathbb{Q}$) gives the functor of points $h_X : \delta\text{- Field} \to \text{Set}$ determined by $k \mapsto X(k)$. We shall show that $\mathbb{C}(n, d, h, g, m)$ is representable, meaning that there is a differentially constructible set which we shall call $\delta\text{-Chow}(n, d, h, g, m)$ and a natural isomorphism between the functors $\mathbb{C}(n, d, h, g, m)$ and $h_{\delta\text{-Chow}(n,d,h,g,m)}$. In fact, we prove a little more in that we produce a universal family of cycles over $\delta\text{-Chow}(n, d, h, g, m)$ so that the natural transformation from $\delta\text{-Chow}(n, d, h, g, m)$ to $\mathbb{C}(n, d, h, g, m)$ is given by taking fibers of this family.

The construction of differential Chow varieties is related to *canonical parameters* in the sense of model theory. In the theory of differentially closed fields, canonical parameters manifest themselves as the generators of fields of definition of differential varieties. In recent years, detailed analyses of canonical parameters have been undertaken in analogy with results of Campana [1] and Fujiki [11] from compact complex manifolds (for instance, see [34, 2, 30]). The following is essentially pointed out by Pillay and Ziegler [34]. Let K be a differential field and x an n-tuple of elements from some differential field extension. Let X be the differential locus of x over K. Let L be a differential field extension of K and let Z be the differential locus of x over L. Let b be a generator for the differential field of definition of Z over K. That is, there is some differential algebraic subvariety $Y \subseteq \mathbb{A}^n \times \mathbb{A}^m$ so that $b \in \mathbb{A}^m(L)$, $(x, b) \in Y_b$ and the second projection map $\pi : Y \to \mathbb{A}^m$ is differentially birational over its image. Consider $\mathcal{Y} := {}_xY$, the fiber of Y over x via the first projection $Y \to \mathbb{A}^n$, which is subvariety of a certain differential Chow variety of X in the sense of this paper. The main result of [34] is that $\mathcal{Y}$ is internal to the constants. One could certainly expand upon these observations to give statements about the structure of differential Chow varieties, but we will not pursue these matters in further detail in this paper.

In [34, page 581], Pillay and Ziegler write of the above situation,

> We are unaware of any systematic development of machinery and language (such as "differential Hilbert spaces") in differential algebraic geometry which is adequate for the geometric translation above. This is among the reasons why we will stick with the language of model theory in our proofs below. The issue of algebraizing the content and proofs is a serious one which will be considered in future papers.

Subsequent work by Moosa and Scanlon [32] did algebraize and generalize much of the work by Pillay and Ziegler, but no systematic development of differential Hilbert schemes or differential Chow varieties appears to have occurred in the decade following Pillay and Ziegler's work. One should view [12] as the beginning of such a systematic development, where the theory of the differential Chow form was developed, and the existence of differential Chow varieties was established in certain very special cases. In [12, section 5], the authors write that they are unable to prove the existence of the differential Chow variety in general. The work here is an extension of [12], in which we will establish the existence of the differential Chow variety in general, answering the most natural question left open by [12]. As we have pointed out above, our general technique is also the descendant of a line thinking that originated (at least in the model theoretic context) with Pillay and Ziegler's work on jet spaces and the linearization of differential equations.

While the theory of canonical parameters provides an ad hoc solution to the problem of parametrizing the differential subvarieties of a given affine space, the theory of differential Chow varieties is superior in some respects. Firstly, the theory of differential Chow varieties provides a natural stratification of the parameter spaces via the discrete index invariant. Secondly, the conceptual definition of the differential Chow varieties through the notion of a differential Chow form permits one to effectively compute the differential Chow coordinates of a differential variety. Indeed, this is done in detail for prime differential ideals in [29]. To compute a canonical parameter using elimination of imaginaries relative to the theory of differentially closed fields of characteristic zero would require an appeal to algebraic invariant theory which itself could be made only after computing nontrivial bounds on the order of the generators of the eventual quotient. Even after this process is completed, the family of varieties one obtains is not characterized by differential algebraic invariants; this is not the case with the differential Chow varieties of this work, which *are* characterized by specifying natural differential algebraic invariants. On the other hand, there is a cost to working with differential Chow coordinates: one must prove that the Chow coordinates of differential varieties with a given index actually form a differentially constructible set; this is precisely what we do in this paper.

The rest of the paper is organized as follows. In section 2, we give background definitions and some preliminary results which we use later in the paper. In addition, we describe the relationship of the problems we consider to the Ritt problem. Following this interlude, we prove the results which eventually allow us to work around the issues involved in the Ritt problem (whose solution would allow for a simplification of the proofs of the results in this paper). In section 3, we describe the necessary background from the classical theory of Chow varieties and introduce the theory of differential Chow forms. In section 4, we establish various bounds on the order and degree of the varieties we consider using the theory of differential Chow forms. In section 5, we establish the existence of differential Chow varieties, proving the main result of the paper.

The appendix gives elementary proofs of several facts from algebraic geometry which we require. The facts proved in the appendix are well known and are frequently used in model theory (for instance, see the citation in appendix 3.1 of [20]), however, several of the proofs in the appendix seem to be new. Constructive proofs of the results in the appendix (which give additional information about certain bounds, rather than simply proving that a certain bound exists) are much more involved (see [37], which corrected and modernized some of the proofs given in [17]). Various other non-constructive proofs of the theorem are given in the literature [13, 15.5.3] [39, where a nonstandard approach is taken] [19, where a model theoretic approach is taken to give an elementary proof].

2. Preliminaries and Prolongations

We fix $\mathcal{U}$ a saturated differentially closed field with a single derivation, δ. Implicitly, all differential fields we consider are subfields of $\mathcal{U}$. If V is an algebraic variety or a differential variety, then an expression of the form "$a \in V$" is shorthand for "$a \in V(\mathcal{U})$". Throughout, K will be a small differential subfield of $\mathcal{U}$ and δ denotes the distinguished derivation on K, $\mathcal{U}$, or, indeed, any differential ring that we consider. Unless explicitly stated to the contrary, all varieties and differential varieties are defined over K and have coordinates in $\mathcal{U}$. By convention, $\mathbb{A}^n$ and $\mathbb{A}^n(\mathcal{U})$ stand for the affine space with coordinates in $\mathcal{U}$.

If $f : X \to Y$ is a morphism of varieties, then by $f(X)$ we mean the scheme theoretic image of X under f. That is, $f(X)$ is the smallest subvariety Z of Y for which f factors through the inclusion $Z \hookrightarrow Y$. On points, $f(X)$ is the Zariski closure of $\{f(a) : a \in X(\mathcal{U})\}$.

We write $K\{x_1, \dots, x_n\}$ for the differential polynomial ring in the variables $x_1, \dots, x_n$ over K. For $m \in \mathbb{N}$, we write

$$K\{x_1, \dots, x_n\}_{\leq m} = K[x_i^{(j)} \ : \ 1 \leq i \leq n, 0 \leq j \leq m]$$

for the subring of differential polynomials of order at most m where we have written $x_i^{(j)}$ for $\delta^j x_i$. We also use $x_i^{[m]}$ to denote the set $\{x_i^{(j)} : j = 0, 1, \dots, m\}$ and sometimes denote $K\{x_1, \dots, x_n\}_{\leq m} = K[x_1^{[m]}, \dots, x_n^{[m]}]$.

If $\mathcal{I} \subseteq K\{x_1, \ldots, x_n\}$ is a differential ideal, then we write $\mathbb{V}(\mathcal{I})$ for the differential subvariety of $\mathbb{A}^n(\mathcal{U})$ defined by the vanishing of all $f \in \mathcal{I}$, and for $S \subseteq \mathbb{A}^n(\mathcal{U})$, we let $\mathbb{I}(S) \subseteq K\{x_1, \ldots, x_n\}$ be the differential ideal of all differential polynomials over K vanishing on S. On the other hand, if $I \subseteq K[x_1, \ldots, x_n]$ is an ideal, then we write $V(I)$ for the variety defined by the vanishing of all $f \in I$, and for $S \subseteq \mathbb{A}^n(\mathcal{U})$, we write $I(S)$ for the ideal of polynomials in $K[x_1, \ldots, x_n]$ which vanish on S.

In general, if R is a commutative ring, we write $\mathcal{Q}(R)$ for its total ring of fractions. When R is a differential ring, so is $\mathcal{Q}(R)$. For a differential variety V, we write $K\langle V\rangle$ for $\mathcal{Q}(K\{x_1, \ldots, x_n\}/\mathbb{I}(V))$. When $\mathbb{I}(V)$ is prime, that is, when V is irreducible, this is called the differential function field of V. For $S \subseteq K\{x_1, \ldots, x_n\}$ we write (S) for the ideal generated by S and $[S]$ for the *differential* ideal generated by S. When $S = \{f\}$ is a singleton, we write $(f) := (S)$ and $[f] := [S]$. Likewise, we write $\mathbb{V}(f)$ for $\mathbb{V}([f])$.

We sometimes speak about "generic points". These should be understood in the sense of Weil-style algebraic (or differential algebraic) geometry. That is, if V is a variety (respectively, differential algebraic variety) over K, then $\eta \in V(\mathcal{U})$ is generic if there is no proper subvariety (respectively, differential subvariety) $W \subsetneq V$ defined over K with $\eta \in W(\mathcal{U})$. Provided that V is irreducible, this is equivalent to asking that the field $K(\eta)$ (respectively, differential field $K\langle\eta\rangle$) be isomorphic over K to $K(V)$ (respectively, $K\langle V\rangle$). We will also say that a point is a generic point of some ideal (or differential ideal) if it is a generic point of the corresponding variety (or differential variety).

2.1. **Methods of algebraic and differential characteristic sets.** In this paper, the Wu-Ritt characteristic set method is a basic tool for establishing a correspondence between differential algebraic cycles and algebraic cycles satisfying certain conditions. In this section we recall the definition and basic properties of algebraic and differential characteristic sets [40, 35].

First, we introduce the algebraic characteristic set method. Consider the polynomial ring $K[x_1, \ldots, x_n]$ and fix an ordering on $x_1, \ldots, x_n$, say, $x_1 < \cdots < x_n$. Given $f \in K[x_1, \ldots, x_n]\backslash K$, the *leading variable* of f is the greatest variable x_k effectively appearing in f, denoted by $\mathrm{lv}(f)$. Regarding f as a univariate polynomial in $\mathrm{lv}(f)$, the leading coefficient of f is called the *initial* of f, denoted by $\mathrm{init}(f)$. A sequence of polynomials $\langle A_1, \ldots, A_r\rangle$ is said to be an *ascending chain*, if either

(1) $r = 1$ and $A_1 \neq 0$, or
(2) all the A_i are nonconstant, $\mathrm{lv}(A_i) < \mathrm{lv}(A_j)$ for $1 \leq i < j \leq r$ and $\deg(A_k, \mathrm{lv}(A_k)) > \deg(A_m, \mathrm{lv}(A_k))$ for $m > k$. (Here $\deg(f, y)$ denotes the degree of f regarded as a polynomial in the variable y.)

Given two polynomials f and g, f is said to be of higher rank than g and denote $f > g$, if either $\mathrm{lv}(f) > \mathrm{lv}(g)$, or $x = \mathrm{lv}(f) = \mathrm{lv}(g)$ and $\deg(f, x) > \deg(g, x)$. If $\mathrm{lv}(f) = \mathrm{lv}(g) = x$ and $\deg(f, x) = \deg(g, x)$, then we say f and g have the same rank. Suppose $\mathcal{A} = \langle A_1, \ldots, A_r\rangle$ and $\mathcal{B} = \langle B_1, \ldots, B_s\rangle$ are two ascending chains in $K[x_1, \ldots, x_n]$. We say $\mathcal{A}$ is of lower rank than $\mathcal{B}$, denoted by $\mathcal{A} \prec \mathcal{B}$, if either

(1) there exists $k \leq \min\{r, s\}$ such that for $i < k$, A_i and B_i have the same rank, and $A_k < B_k$, or
(2) $r > s$, and for each $i \leq s$, A_i and B_i have the same rank.

Given an ideal $\mathcal{I}$ in $K[x_1, \ldots, x_n]$, an ascending chain contained in $\mathcal{I}$ which is of lowest rank is called an algebraic characteristic set of $\mathcal{I}$. If $V \subseteq \mathbb{A}^n$ is an irreducible variety, then an algebraic characteristic set of V is defined as a characteristic set of its corresponding prime ideal $I(V)$.

Let $\mathcal{A} = \langle A_1, A_2, \ldots, A_t \rangle$ be an ascending chain. We call $\mathcal{A}$ an *irreducible ascending chain* if for any $1 \leq i \leq t$, there can not exist any relation of the form

$$T_i A_i = B_i C_i, \quad \bmod\ (A_1, \ldots, A_{i-1})$$

where B_i, C_i are polynomials with the same leader as A_i, T_i is a polynomial with lower leader than A_i, and B_i, C_i, T_i are reduced with respect to $A_1, \ldots, A_{i-1}$ ([40]). In other words, an ascending chain $\mathcal{A}$ is irreducible if and only if there exist no polynomials P and Q which are reduced with respect to $\mathcal{A}$ and $PQ \in \mathrm{asat}(\mathcal{A}) = (\mathcal{A}) : I_{\mathcal{A}}^{\infty}$, where $I_{\mathcal{A}}^{\infty}$ stands for the set of all products of powers of $\mathrm{init}(A_i)$. By [35, p.89], for an ascending chain $\mathcal{A}$ to be a characteristic set of a prime polynomial ideal, it is necessary and sufficient that $\mathcal{A}$ is irreducible.

We now return to ordinary differential polynomial algebra and introduce differential characteristic methods for differential polynomials. Fix a sequence of differential variables $x_1, x_2, \ldots, x_n$ and consider the differential polynomial ring $K\{x_1, \ldots, x_n\}$. A differential ranking is a total order $\prec$ on the set $\Theta := \{x_i^{(j)} : i \leq n, j \in \mathbb{N}\}$ satisfying

- For all $\theta \in \Theta$, $\delta\theta \succ \theta$ and
- If $\theta_1 \succ \theta_2$, then $\delta\theta_1 \succ \delta\theta_2$.

An orderly ranking is a differential ranking which satisfies in addition

- If $k > \ell$, then $\delta^k x_i \succ \delta^\ell x_j$ for all i and j.

Throughout the paper, we fix some orderly ranking $\mathscr{R}$, and when we talk about characteristic set methods in the polynomial ring $K\{x_1, \ldots, x_n\}_{\leq \ell}$, the ordering on $(x_i^{(j)})_{1 \leq i \leq n; j \leq \ell}$ induced by $\mathscr{R}$ is fixed.

Let f be a differential polynomial in $K\{x_1, \ldots, x_n\}$. The *leader* of f, denoted by $\mathrm{ld}(f)$, is the greatest $v \in \Theta$ with respect to $\prec$ which appears effectively in f. Regarding f as a univariate polynomial in $\mathrm{ld}(f)$, its leading coefficient is called the *initial* of f, denoted by $\mathrm{init}(f)$, and the partial derivative of f with respect to $\mathrm{ld}(f)$ is called the *separant* of f, denoted by S_f. For any two differential polynomials f, g in $K\{x_1, \ldots, x_n\}$, f is said to be of lower rank than g, denoted by $f < g$, if

- $\mathrm{ld}(f) \prec \mathrm{ld}(g)$ or
- $\mathrm{ld}(f) = \mathrm{ld}(g)$ and $\deg(f, \mathrm{ld}(f)) < \deg(g, \mathrm{ld}(f))$.

The differential polynomial f is said to be reduced with respect to g if no proper derivative of $\mathrm{ld}(g)$ appears in f and $\deg(f, \mathrm{ld}(g)) < \deg(g, \mathrm{ld}(g))$.

Let $\mathcal{A}$ be a set of differential polynomials. Then $\mathcal{A}$ is said to be an auto-reduced set if each differential polynomial in $\mathcal{A}$ is reduced with respect to any other element of $\mathcal{A}$. Every auto-reduced set is finite [35].

Let $\mathcal{A}$ be an auto-reduced set. We denote by $\mathrm{H}_{\mathcal{A}}$ the set of all initials and separants of $\mathcal{A}$ and by $\mathrm{H}_{\mathcal{A}}^{\infty}$ the minimal multiplicative set containing $\mathrm{H}_{\mathcal{A}}$. The saturation differential ideal of $\mathcal{A}$ is defined to be

$$\mathrm{sat}(\mathcal{A}) = [\mathcal{A}] : \mathrm{H}_{\mathcal{A}}^{\infty} = \{f \in K\{x_1, \ldots, x_n\} \ : \ \exists h \in \mathrm{H}_{\mathcal{A}}^{\infty} \text{ for which } hf \in [\mathcal{A}]\} \ .$$

An auto-reduced set $\mathcal{C}$ contained in a differential polynomial set $\mathcal{S}$ is said to be a *characteristic set* of $\mathcal{S}$ if $\mathcal{S}$ does not contain any nonzero element reduced with respect to $\mathcal{C}$. A characteristic set $\mathcal{C}$ of a differential ideal $\mathcal{J}$ reduces all elements of $\mathcal{J}$ to zero. Furthermore, if $\mathcal{J}$ is prime, then $\mathcal{J} = \mathrm{sat}(\mathcal{C})$.

We can define an auto-reduced set to be *irreducible* if when considered as an algebraic ascending chain in the underlying polynomial ring, it is irreducible. We have ([35, p.107])

Lemma 2.1. *Let $\mathcal{A}$ be an auto-reduced set. Then a necessary and sufficient condition for $\mathcal{A}$ to be a characteristic set of a prime differential ideal (or an irreducible differential variety) is that $\mathcal{A}$ is irreducible. Moreover, in the case $\mathcal{A}$ is irreducible,* $\mathrm{sat}(\mathcal{A})$ *is prime and $\mathcal{A}$ is a differential characteristic set of it.*

Remark 2.2. Differential characteristic sets and algebraic characteristic sets have the following obvious relation: Suppose $\mathcal{A}$ is a differential characteristic set of a prime differential ideal $\mathcal{I} \subset K\{x_1, \dots, x_n\}$ under a fixed orderly ranking $\mathscr{R}$. Then for any $\ell \in \mathbb{N}$, $\{\delta^k f : f \in \mathcal{A}, \mathrm{ord}(f) \leq \ell, k \leq \ell - \mathrm{ord}(f)\}$ is an algebraic characteristic set of the prime ideal $\mathcal{I} \cap K\{x_1, \dots, x_n\}_{\leq \ell}$ under the ordering induced by $\mathscr{R}$.

Let $\mathcal{I}$ be a prime differential ideal in $K\{x_1, \dots, x_n\}$. The *differential dimension* of $\mathcal{I}$ is defined as the differential transcendence degree of the differential extension field $\mathcal{Q}(K\{x_1, \dots, x_n\}/\mathcal{I})$ over K, denoted by δ-$\dim(\mathcal{I})$. As a differential invariant, the differential dimension of $\mathcal{I}$ can be read off from its Kolchin polynomial, which can characterize the size of $\mathbb{V}(\mathcal{I})$.

Definition 2.3. [22] Let $\mathcal{I}$ be a prime differential ideal of $K\{x_1, \dots, x_n\}$. Then there exists a unique numerical polynomial $\omega_{\mathcal{I}}(t)$ such that

$$\omega_{\mathcal{I}}(t) = \mathrm{tr.deg}\ \mathcal{Q}\big(K\{x_1, \dots, x_n\}_{\leq t}/(\mathcal{I} \cap K\{x_1, \dots, x_n\}_{\leq t})\big)/K$$

for all sufficiently large $t \in \mathbb{N}$. The polynomial $\omega_{\mathcal{I}}(t)$ is called the *Kolchin polynomial* of $\mathcal{I}$ or its corresponding irreducible differential variety.

Lemma 2.4. [36, Theorem 13] *Let $\mathcal{I}$ be a prime differential ideal in $K\{x_1, \dots, x_n\}$ of dimension d. Then the Kolchin polynomial of $\mathcal{I}$ has the form $\omega_{\mathcal{I}}(t) = d(t+1)+h$, where h is defined to be the* order *of $\mathcal{I}$ or of $\mathbb{V}(\mathcal{I})$, that is,* $\mathrm{ord}(\mathcal{I}) = h$. *Let $\mathcal{A}$ be a characteristic set of $\mathcal{I}$ under any orderly ranking. Then,* $\mathrm{ord}(\mathcal{I}) = \sum_{f \in \mathcal{A}} \mathrm{ord}(f)$ *and $d = n - \mathrm{card}(\mathcal{A})$.*

Recall that the set of numerical polynomials can be totally ordered with respect to the ordering: $\omega_1 \leq \omega_2$ if and only if $\omega_1(s) \leq \omega_2(s)$ for all sufficiently large $s \in \mathbb{N}$. Given a differential variety W, we define a *generic component* of V to be an irreducible component which has maximal Kolchin polynomial among all the components of W. Lemma 2.4 defines order for prime differential ideals or its corresponding irreducible differential varieties. In this paper, we sometimes talk about the order of an arbitrary differential variety where we actually mean the order of its generic components.

2.2. **Prolongation sequences and prolongation admissible varieties.** We follow the notation of section 2 of [31]. There, the authors define a sequence of functors τ_m indexed by the natural numbers from varieties over K to varieties over K (to be honest, the functor may return a nonreduced scheme, but the distinction between a scheme and its reduced subscheme is immaterial here). For affine space itself, one has $\tau_m(\mathbb{A}^n) \cong \mathbb{A}^{n(m+1)}$ where if we present $\mathbb{A}^n$ as $\mathrm{Spec}(K[x_1, \dots, x_n])$, then $\tau_m(\mathbb{A}^n) = \mathrm{Spec}(K\{x_1, \dots, x_n\}_{\leq m})$. If $V \subseteq \mathbb{A}^m$ is a subvariety of affine space, then $\tau_m V = \mathrm{Spec}\left(K\{x_1, \dots, x_n\}_{\leq m}\big/(\{\delta^j f \ : \ f \in I(V), j \leq m\})\right)$. Note that the ideal $(\{\delta^j f \ : \ f \in I(V), j \leq m\})$ is contained in $\mathbb{I}(V) \cap K\{x_1, \dots, x_n\}_{\leq m}$, but the inclusion may be proper.

There is a natural differential algebraic map $\nabla_m : V \to \tau_m V$ given on points valued in a differential ring by

$$(a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n; \delta(a_1), \ldots, \delta(a_n); \ldots; \delta^m(a_1), \ldots, \delta^m(a_n)) \ .$$

We call points in the image of ∇_m *differential points.*

The image of the map ∇_m need not be Zariski dense, even on $\mathcal{U}$-valued points. For any differential subvariety $W \subseteq \mathbb{A}^n$, we define $B_m(W)$ to be the Zariski closure in $\tau_m \mathbb{A}^n$ of $\nabla_m(W)$.

The functors τ_m form a projective system with the natural transformation $\pi_{m,\ell} : \tau_m \to \tau_\ell$ for $\ell \leq m$ given by projecting onto the coordinates corresponding to the first ℓ derivatives. We write $\pi_{m,\ell} : \tau_m V \to \tau_\ell V$ rather than $\pi^V_{m,\ell}$. Moreover, τ_0 is simply the identity functor so that we write V rather than $\tau_0(V)$. From the definition, for $W \subseteq \mathbb{A}^n$ a differential subvariety, it is clear that $\pi_{m,\ell}$ restricts to make the sequence of varieties $(B_m(W))_{m=0}^\infty$ into a projective system of algebraic varieties in which each map in the system is dominant.

We write τ^m for the result of composing the functor τ_1 with itself m times. There is a natural transformation $\rho_m : \tau_m \to \tau^m$ which for any algebraic variety V gives a closed embedding $\rho_m : \tau_m V \hookrightarrow \tau^m V$. To ease notation, let us write the map ρ in coordinates only for the case of $V = \mathbb{A}^1$ and $m = 3$. The general case requires one to decorate the variables with further subscripts and to nest the coordinates more deeply. Here

$$\rho_3(x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)}) = (((x^{(0)}, x^{(1)}), (x^{(1)}, x^{(2)})), ((x^{(1)}, x^{(2)}), (x^{(2)}, x^{(3)})))$$

For the formal definitions of the items discussed above, please refer to [32, 31].

Definition 2.5. A sequence of varieties $X_\ell \subseteq \tau_\ell \mathbb{A}^n$ ($\ell \in \mathbb{N}$) is called a *prolongation sequence* if

(1) The map $X_{\ell+1} \to X_\ell$ induced by the projection map $\pi_{\ell+1,\ell} : \tau_{\ell+1}\mathbb{A}^n \to \tau_\ell \mathbb{A}^n$ is dominant.
(2) For all ℓ, $\rho_{\ell+1}(X_{\ell+1})$ is a closed subvariety of $\tau_1(\rho_\ell(X_\ell))$.

Given a sequence of algebraic varieties $(X_\ell)_{\ell \geq 0}$ with $X_\ell \subseteq \tau_\ell \mathbb{A}^n$, the differential variety V corresponding to $(X_\ell)_{\ell \geq 0}$ is

$$V = \{b \in \mathbb{A}^n : (\forall \ell) \ \nabla_\ell(b) \in X_\ell\} \ .$$

From the point of view of differential ideals, if $\mathcal{I} = \bigcup_{\ell=0}^\infty I(X_\ell) \subseteq K\{x_1, \ldots, x_n\}$, then $V = \mathbb{V}(\mathcal{I})$.

By Definition 2.5, a sequence of varieties $(X_\ell \subset \tau_\ell \mathbb{A}^n)_{\ell \geq 0}$ is a prolongation sequence if and only if for each $\ell \geq 0$, $X_\ell = B_\ell(V)$ where V is the differential variety corresponding to $(X_\ell)_{\ell \geq 0}$.

There is a bijective correspondence between irreducible prolongation sequences (by which we mean each variety in the sequence is irreducible) and affine irreducible differential varieties. Given a differential variety V, the prolongation sequence corresponding to V is given by $X_\ell = B_\ell(V)$ for all $\ell \geq 0$ [31, the discussion preceding Definition 2.8]. Thus, prolongation sequences are in one-to-one correspondence with affine differential algebraic varieties, and the Noetherianity of the Kolchin topology guarantees that a finite portion of a prolongation sequence determines the entire sequence.

The varieties which appear in a prolongation sequence are interesting to us. Below, we first study the irreducible ones and define prolongation admissibility for irreducible varieties.

Definition 2.6. Given an irreducible algebraic variety $V \subseteq \tau_\ell \mathbb{A}^n$, we say that V is prolongation admissible if $\rho_\ell(V) \subseteq \tau(\rho_{\ell-1}(\pi_{\ell,\ell-1}(V)))$.

For irreducible prolongation admissible varieties, it is easy to establish the stronger fact that $\rho_\ell(V) \subseteq \tau^{\ell-d}(\rho_d(\pi_{\ell,d}(V)))$ for all $0 \leq d < \ell$. Indeed, we have natural transformations $\rho_\ell \to \rho_{\ell-d}\pi_{\ell,\ell-d}$ for $0 < d \leq \ell$ given by forgetting the coordinates corresponding to the k^{th} derivatives for $\ell - d < k \leq \ell$. Applying τ and pre-composing with $\pi_{\ell,\ell-1}$ to $\rho_{\ell-1} \to \rho_{\ell-d}\pi_{\ell-1,\ell-d}$ we obtain natural transformations $\tau\rho_{\ell-1}\pi_{\ell,\ell-1} \to \tau\rho_{\ell-d}\pi_{\ell,\ell-d}$. Thus, from an inclusion $\rho_\ell(V) \subseteq \tau\rho_{\ell-1}\pi_{\ell,\ell-1}(V)$ we deduce that we must have an inclusion $\rho_{\ell-d+1}\pi_{\ell,\ell-d+1}(V) \subseteq \tau\rho_{\ell-d}\pi_{\ell,\ell-d}(V)$ as we see from the following diagram:

$$\begin{CD} \rho_\ell(V) @>>> \tau\rho_{\ell-1}(\pi_{\ell,\ell-1}(V)) \\ @VVV @VVV \\ \rho_{\ell-d+1}(\pi_{\ell,\ell-d+1}(V)) @>>> \tau\rho_{\ell-d}(\pi_{\ell,\ell-d}(V)). \end{CD}$$

Using the fact that τ preserves inclusion, and iterating, we have

$$\rho_\ell(V) \subseteq \tau\rho_{\ell-1}(\pi_{\ell,\ell-1}(V)) \subseteq \tau^2\rho_{\ell-2}(\pi_{\ell,\ell-2}(V)) \subseteq \cdots \subseteq \tau^d\rho_{\ell-d}(\pi_{\ell,\ell-d}(V))$$

as claimed.

The following fact is the basis of the well known geometric axioms for differentially closed fields, written in our language:

Fact 2.7. *Given irreducible varieties V and W of $\mathbb{A}^n(\mathcal{U})$ with $W \subseteq \tau_1(V)$ so that the restriction of $\pi_{1,0}$ to W is a dominant map to V, then for any $U \subseteq W(\mathcal{U})$, a nonempty Zariski open set, there is $a \in V(\mathcal{U})$ such that $\nabla_1(a) \in U$.*

Indeed, Fact 2.7 characterizes differentially closed fields amongst algebraically closed differential fields of characteristic zero; for details, see [33].

Lemma 2.8. *Suppose that $V \subseteq \tau_\ell(\mathbb{A}^n)$ is an irreducible prolongation admissible variety. Then for any nonempty open subset $U \subseteq V$, there is some $a \in \mathbb{A}^n(\mathcal{U})$ such that $\nabla_\ell(a) \in U$.*

Proof. Since V is prolongation admissible, $\rho_\ell(V) \subseteq \tau_1(\rho_{\ell-1}(\pi_{\ell,\ell-1}(V)))$. So, we have the following commutative diagram:

$$\begin{CD} V @>{\rho_\ell}>> \rho_\ell(V) \subseteq \tau_1(\rho_{\ell-1}(\pi_{\ell,\ell-1}(V))) \\ @V{\pi_{\ell,\ell-1}}VV @VV{\pi_{1,0}}V \\ \pi_{\ell,\ell-1}(V) @>>{\rho_{\ell-1}}> \rho_{\ell-1}(\pi_{\ell,\ell-1}(V)). \end{CD}$$

Since each ρ is an embedding, we must have that the restriction of $\pi_{1,0}$ maps $\rho_\ell(V)$ dominantly to $\rho_{\ell-1}(\pi_{\ell,\ell-1}(V))$. Thus, by Fact 2.7, the set of points

$$\{\nabla_1(a) | a \in \rho_{\ell-1}(\pi_{\ell,\ell-1}(V)),\ \nabla_1(a) \in \rho_\ell(V)\}$$

is Zariski dense in $\rho_\ell(V)$, so the set

$$\rho_\ell^{-1}(\{\nabla_1(a) | a \in \rho_{\ell-1}(\pi_{\ell,\ell-1}(V)),\ \nabla_1(a) \in \rho_\ell(V)\})$$

is Zariski dense in V. Every such point has the form $\nabla_\ell(b)$ for $b \in \mathbb{A}^n$, proving the claim. □

Remark 2.9. From the proof of Lemma 2.8, an irreducible algebraic variety is prolongation admissible if and only if the differential points form a dense subset. For an arbitrary algebraic variety, we define it to be *prolongation admissible* if all of its irreducible components are prolongation admissible. In other words, an algebraic variety is prolongation admissible if and only if the differential points form a dense subset. So every variety in a prolongation sequence is prolongation admissible. Furthermore, prolongation admissible varieties are precisely the varieties which appear as elements of a prolongation sequence.

Given two prolongation sequences $X := (X_\ell \subset \tau_\ell \mathbb{A}^n)_{\ell=0}^\infty$ and $Y := (Y_\ell \subset \tau_\ell \mathbb{A}^n)_{\ell=0}^\infty$, define $X \leq Y$ if and only if $X_\ell \subseteq Y_\ell$ for each $\ell \geq 0$. With this binary relation, the set of all prolongation sequences forms a partially order set. From the definition, it is easy to see that if $\mathcal{V}$ is a set of prolongation sequences $X := (X_\ell \subseteq \tau_\ell \mathbb{A}^n)_{\ell=0}^\infty$, then the sequence $(\overline{\bigcup_{X\in\mathcal{V}} X_\ell})_{\ell=0}^\infty$ is also a prolongation sequence. This justifies the following definition.

Definition 2.10. Given a variety $V \subseteq \tau_h(\mathbb{A}^n)$, the prolongation sequence generated by V is the maximal prolongation sequence $(V_\ell \subseteq \tau_\ell \mathbb{A}^n)_{\ell \geq 0}$ such that $V_h \subseteq V$.

The following lemma follows from the observation above that the closure of an arbitrary union of prolongation admissible varieties is also prolongation admissible.

Lemma 2.11. *Given $V \subseteq \tau_h \mathbb{A}^n$, there is a finite set of irreducible maximal prolongation admissible subvarieties of V.*

Proof. Let $\mathcal{W}$ be the set of all prolongation admissible subvarieties of V and set $W = \overline{\cup \mathcal{W}}$. Then clearly, $\overline{\{\nabla(a) : \nabla(a) \in W\}} = W$. Let $W = \cup_{i=1}^m W_i$ be the irredundant irreducible decomposition of W. Since $\overline{\cup_{i=1}^m \{\nabla(a) : \nabla(a) \in W_i\}} = W$, $\overline{\{\nabla(a) : \nabla(a) \in W_i\}} = W_i$ for each i. Thus, W_i is prolongation admissible and these W_i are the only maximal irreducible prolongation admissible subvarieties of V. □

Lemma 2.12. *Given a prolongation admissible variety $V \subseteq \tau_h(\mathbb{A}^n) = \mathbb{A}^{n(h+1)}$, the prolongation sequence generated by V, denoted by $(V_i)_{i\in\mathbb{N}}$, has the property that $V_h = V$ and for each i,*

$$V_i = \overline{\{(a, \delta a, \ldots, \delta^i a) \,|\, a \in \mathcal{U},\ (a, \delta a, \ldots \delta^h a) \in V\}}.$$

Proof. It follows from Definition 2.5 that $(V_i)_{i\in\mathbb{N}}$ forms a prolongation sequence. It is easy to see that $(V_i)_{i\in\mathbb{N}}$ is maximal among all prolongation sequences $(W_i)_{i\in\mathbb{N}}$ with $W_h \subseteq V$. And by Remark 2.9, $V_h = V$. □

Irreducible prolongation admissible varieties are of special interest in this paper. The following lemma shows that the algebraic characteristic sets of irreducible prolongation admissible varieties have a special form.

Lemma 2.13. *Let $V \subset \tau_h \mathbb{A}^n$ be an irreducible prolongation admissible variety and $\mathcal{A}$ a characteristic set of V under the standard orderly ranking. Rewrite $\mathcal{A}$ in the* *following form*

$$\mathcal{A} = \begin{array}{lll} \langle A_{10}, & \ldots, & A_{1\ell_1} \\ \ldots & \ldots & \ldots \\ A_{p0}, & \ldots, & A_{p\ell_p}\rangle \end{array} \tag{1}$$

such that $\mathrm{lv}(A_{ij}) = x_{\sigma(i)}^{(o_{ij})}$ $(j = 0, \ldots, \ell_i)$ *and* $o_{i0} < o_{i1} < \cdots < o_{i\ell_i}$. *Then for each* i, $\ell_i = h - o_{i0}$, *and if* $\ell_i > 1$, *then for all* $k = 1, \ldots, \ell_i$, $o_{ik} = o_{i0} + k$ *and* A_{ik} *is linear in* $x_{\sigma(i)}^{(o_{i0}+k)}$. *Moreover,* $\mathcal{A}$ *is a consistent differential system.*

Proof. Let W be the differential variety associated to the prolongation sequence generated by V. Suppose $W = \bigcup_i W_i$ is an irredundant irreducible decomposition of W. Then $B_h(W) = \bigcup_i B_h(W_i) = V$. Since V is irreducible, there exists i_0 such that $B_h(W_{i_0}) = V$. Let $\mathcal{B}^\delta := \langle B_1, B_2, \ldots, B_m\rangle$ be a differential characteristic set of W_{i_0} and suppose for $i \le p$, $s_i = \mathrm{ord}(B_i) \le h$ and for $i > p$, $\mathrm{ord}(B_i) > h$. Then by Remark 2.2, $\langle B_1, B_1', \ldots, B_1^{(h-s_1)}, \ldots, B_p, B_p', \ldots, B_p^{(h-s_p)}\rangle$ is an algebraic characteristic set of V. Since any two characteristic sets of V have the same rank, $\mathcal{A}$ has the desired form described as above. □

Definition 2.14. Let V be an irreducible prolongation admissible variety of $\tau_h(\mathbb{A}^n)$ and W be the differential variety corresponding to the prolongation sequence generated by V. A component W_1 of W is called a *dominant* component if it satisfies $B_h(W_1) = V$.

Lemma 2.15. *Let* V *be an irreducible prolongation admissible subvariety of* $\tau_h(\mathbb{A}^n)$. *Let* W *be the differential variety corresponding to the prolongation sequence generated by* V. *Then there is a unique dominant component. Moreover, the Kolchin polynomial of the dominant component* W_1 *has the form*

$$\omega_{W_1}(t) = \big(\dim(V) - \dim(\overline{\pi_{h,h-1}(V)})\big)(t-h) + \dim(V),$$

and for $t \ge h-1$, $\dim(B_t(W_1)) = \omega_{W_1}(t)$.

Proof. Since V is prolongation admissible, by Lemma 2.13, V has a characteristic set $\mathcal{A}$ of the form

$$\mathcal{A} = \begin{array}{lll} \langle A_{10}, & \ldots, & A_{1\ell_1} \\ \ldots & \ldots & \ldots \\ A_{p0}, & \ldots, & A_{p\ell_p}\rangle \end{array}.$$

satisfying the corresponding property in Lemma 2.13.

Let $\mathcal{B}^\delta = \langle A_{10}, A_{20}, \ldots, A_{p0}\rangle$. Then it is clear that $\mathcal{B}^\delta$ is a differential auto-reduced set which is irreducible. Then by Lemmas 2.1 and 2.4, $T = \mathbb{V}(\mathrm{sat}(\mathcal{B}^\delta))$ is an irreducible differential variety of differential dimension $n-p$. We first claim that $B_h(T) = V$. Indeed, since V is prolongation admissible, $I(V) = \mathbb{I}(W) \cap K[(x_i^{(j)})_{1\le i\le n; j\le h}]$. So $A_{i0} \in \mathbb{I}(W)$ implies $A_{i0}^{(\ell)} \in I(V)$ for $\ell \le h - \mathrm{ord}(A_{i0})$. Let $\mathcal{C} := \langle A_{10}, \ldots, A_{10}^{(h-\mathrm{ord}(A_{10}))}, \ldots, A_{p0}, \ldots, A_{p0}^{(h-\mathrm{ord}(A_{p0}))}\rangle$. Then we have $\mathrm{asat}(\mathcal{C}) = \mathrm{sat}(\mathcal{B}^\delta) \cap K[(x_i^{(j)})_{1\le i\le n; j\le h}]$ and correspondingly, $B_h(T) = V(\mathrm{asat}(\mathcal{C}))$. On the other hand, since $\mathcal{C} \subset I(V) = \mathrm{asat}(\mathcal{A})$ and the auto-reduced property of $\mathcal{A}$ implies that the initial and separant of A_{i0} are all reduced with respect to $\mathcal{A}$, $\mathrm{asat}(\mathcal{C}) \subseteq \mathrm{asat}(\mathcal{A})$. Note that $\mathrm{asat}(\mathcal{C})$ and $\mathrm{asat}(\mathcal{A})$ are prime ideals of the same dimension, so we have $\mathrm{asat}(\mathcal{C}) = \mathrm{asat}(\mathcal{A})$. Thus, $B_h(T) = V(\mathrm{asat}(\mathcal{A})) = V$.

We now show that $T \subseteq W$. Let $b \in \mathbb{A}^n$ be a generic point of T. By Lemma 2.12, the prolongation sequence $(V_i)_{i\in\mathbb{N}}$ generated by V has the property that for each i,

$$V_i = \overline{\{(a, \delta a, \dots, \delta^i a) \mid a \in \mathcal{U},\, (a, \delta a, \dots \delta^h a) \in V\}}.$$

So $W = \{a \in \mathbb{A}^n : \nabla_i(a) \in V_i, i \in \mathbb{N}\}$. Since $B_h(T) = V$, $\nabla_h(b) \in V$, $b \in W$. So $T \subseteq W$. It remains to show that T is the unique dominant component of W.

Let $W = \bigcup_i W_i$ be the irredundant irreducible decomposition of W. Since $B_h(W) = V$, $B_h(W_i) \subseteq V$ for each i and there exists at least one component W_i satisfying $B_h(W_i) = V$, that is, a dominant component. Now suppose W_{i_0} is an arbitrary dominant component of W. Then $B_h(W_{i_0}) = V$ and by Remark 2.2, $\mathcal{B}^\delta$ may be a subset of a differential characteristic set of W_{i_0}. So $\mathrm{sat}(\mathcal{B}^\delta) \subseteq \mathbb{I}(W_{i_0})$ and $W_{i_0} \subseteq T$ follows. Since $T \subseteq W$, there exists i_1 such that $T \subseteq W_{i_1}$. From $W_{i_0} \subseteq T \subseteq W_{i_1}$, we have $W_{i_0} = T = W_{i_1}$. Thus, T is the unique dominant component of W.

For the second assertion, since the order of the polynomials in a characteristic set of T under an orderly ranking is bounded by h, $\omega_T(t) = \dim(B_t(T))$ for $t \geq h-1$. Since $\omega_T(t) = \delta\text{-}\dim(T)(t+1) + \mathrm{ord}(T)$, $B_h(T) = V$ and $B_{h-1}(T) = \pi_{h,h-1}(V)$, $\omega_T(t)$ has the desired form. □

Remark 2.16. Given an irreducible prolongation admissible variety $V \subsetneq \tau_h(\mathbb{A}^n)$, Lemma 2.15 tells us that the differential variety W corresponding to the prolongation sequence generated by V has a unique dominant component, W_1. Apart from the above, Lemma 2.15 and its proof give us more information of W_1, which we would like to point out here to make things clearer.

First, both the differential dimension and order of W_1 can be computed from V. To be more specific, the differential dimension of W_1 is equal to $d = \dim(V) - \dim(\overline{\pi_{h,h-1}(V)})$, and the order of W_1 is equal to $\dim(V) - d(h+1)$.

Second, W_1 can be recovered from V in terms of characteristic sets: Let $\mathcal{A} := \langle A_{10}, A_{11}, \dots, A_{1\ell_1}, \dots, A_{p0}, A_{p1}, \dots, A_{p\ell_p}\rangle$ be an algebraic characteristic set of V arranged as in the form (2). Then $\mathcal{B} := \langle A_{10}, A_{20}, \dots, A_{p0}\rangle$ is a differential characteristic set of W_1, and $\mathrm{sat}(\mathcal{B})$ is the defining differential ideal of W_1.

In addition, as we will see in Section 3, the differential Chow form of W_1 and the index of W_1 can be computed from V. The previous result has a partial differential analog, but the situation is more complicated. See [28] for details.

We conclude this section by giving simple examples to illustrate prolongation admissible varieties as well as results in Lemma 2.15 and Remark 2.16.

Example 2.17. Let $V \subset \tau_2\mathbb{A}^2$ be the algebraic variety defined by $x_1' = 0$ and $x_2'' = 0$. Then the prolongation sequence generated by V is $(V_\ell \subseteq \tau_\ell\mathbb{A}^2)_{\ell=0}^\infty$ where $V_0 = \mathbb{A}^2$, $V_1 = V(x_1') \subset \tau_1\mathbb{A}^2$, $V_2 = V(x_1', x_1'', x_2'') \subset \tau_2\mathbb{A}^2$ and $V_\ell = V(x_1', x_1'', \dots, x_1^{(\ell)}, x_2'', \dots, x_2^{(\ell)}) \subseteq \tau_\ell\mathbb{A}^2$ ($\ell \geq 3$). Since $V_2 \subsetneq V$, V is not a prolongation admissible variety.

Let $U = V_2 = V(x_1', x_1'', x_2'') \subset \tau_2\mathbb{A}^2$. Then as illustrated above, U is a prolongation admissible variety and the prolongation sequence generated by U is just the same $(V_\ell \subseteq \tau_\ell\mathbb{A}^2)_{\ell=0}^\infty$ as above. The differential variety corresponding to $(V_\ell \subseteq \tau_\ell\mathbb{A}^2)_{\ell=0}^\infty$ is $W = \mathbb{V}(x_1', x_2'')$, which is irreducible, hence also the dominant component of itself. Note that $U = B_2(W)$.

The following examples show how each irreducible differential variety W of order h can be determined by the corresponding prolongation admissible variety $B_h(W)$.

Example 2.18. Let $W = \mathbb{V}(x_1', x_2'') \subset \mathbb{A}^2$ be the irreducible differential variety defined by $x_1' = 0, x_2'' = 0$. Clearly, the order of W is $h = 3$ and $V = B_3(W) = V(x_1', x_1'', x_1^{(3)}, x_2'', x_2^{(3)}) \subset \tau_3\mathbb{A}^2$ is prolongation admissible. The prolongation sequence generated by V is the same $(V_\ell)_{\ell=0}^\infty$ as in Example 2.17. As illustrated in Example 2.17, then the differential variety corresponding to $(V_\ell)_{\ell=0}^\infty$ is W, which is the unique dominant component of itself.

Example 2.19. Let $W = \mathbb{V}(x'^2 - 4x, x'' - 2) \subset \mathbb{A}^1$. It is clear that W is an irreducible differential variety of order $h = 1$. Moreover, $V = B_1(W) = V(x'^2 - 4x) \subset \tau_1\mathbb{A}^1$ is prolongation admissible and $\mathcal{A} := x'^2 - 4x$ is an algebraic characteristic set of V under the ordering $x < x'$. The prolongation sequence generated by V is $(V_\ell)_{\ell=0}^\infty$ with $V_0 = \mathbb{A}^1$, $V_1 = V$, $V_2 = V(x'^2 - 4x, x'(x'' - 2)) \subset \tau_2\mathbb{A}^1$ and $V_\ell = V([x'^2 - 4x] \cap K[x, x', \ldots, x^{(\ell)}]) \subset \tau_\ell\mathbb{A}^1$ $(\ell \geq 3)$. Then the differential variety corresponding to $(V_\ell)_{\ell=0}^\infty$ is $X = \mathbb{V}(x'^2 - 4x)$, which has two components, the general component W in Ritt's sense and the singular component $\mathbb{V}(x)$. Clearly, W is the unique dominant component of X, which happens to be a generic component of X. Moreover, from the proof of Lemma 2.15, $\mathcal{B}^\delta := \langle x'^2 - 4x \rangle$ is a differential characteristic set of W, so $W = \mathrm{sat}(\mathcal{B}^\delta)$ can be determined by $V = B_1(W)$, which coincides with the classical result.

Generic components of a differential variety are components with maximal Kolchin polynomial. One might think that the differential variety associated to an irreducible prolongation admissible variety also has a unique generic component and the generic component is also the dominant one. Unfortunately, this is invalid. The following example shows how generic components and dominant components may differ.

Example 2.20. Let $n = 3$, $f_1 = y_1''^2 - 4y_1'$, $f_2 = y_1'y_2'' + y_2^2 - 1$, and $f_3 = y_1y_3'' + y_2^2 - 1$. Let $V = V(f_1, f_2, f_3) \subset \tau_2\mathbb{A}^3$. With the help of Maple, we know $I = (f_1, f_2, f_3) \subset K[y_i^{(j)} : j \leq 2]$ is a prime ideal of dimension 6. So $\mathrm{sat}(f_1, f_2, f_3) \cap K[(y_i^{(j)}) : j \leq 3] = I$ and $V = V(I)$ is irreducible and prolongation admissible. By Remark 2.16, the differential variety $W = \mathbb{V}(f_1, f_2, f_3) \subset \mathbb{A}^3$ has a unique dominant component $\mathbb{V}\big(\mathrm{sat}(f_1, f_2, f_3)\big)$.

Performing the Rosenfeld-Groebner algorithm, one can show that W has two generic components, $W_1 = \mathbb{V}(y_1, y_2 - 1)$ and $W_2 = \mathbb{V}(y_1, y_2 + 1)$. Clearly, both $B_2(W_1)$ and $B_2(W_2)$ are properly contained in V, so both of them are not dominant components.

2.3. **Definability in the theory of DCF$_0$ and Ritt problem.** We shall speak of definable families of definable sets and of certain properties being definable in families. These are general notions but we shall use them only for the theories of algebraically closed fields of characteristic zero and of differentially closed fields of characteristic zero. In these cases, "definable" is synonymous with "constructible" or "differentially constructible", respectively.

Definition 2.21. We say that a family of sets $\{X_a\}_{a\in B}$ is a *definable family* if there are formulae $\psi(x; y)$ and $\theta(y)$ so that B is the set of realizations of θ and for each $a \in B$, X_a is the set of realizations of $\psi(x; a)$.

Given a property $\mathcal{P}$ of definable sets, we say that $\mathcal{P}$ is definable in families if for any family of definable sets $\{X_a\}_{a\in B}$ given by the formulae $\psi(x; y)$ and $\theta(y)$, there is a formula $\phi(y)$ so that the set $\{a \in B \,:\, X_a \text{ has property } \mathcal{P}\}$ is defined by ϕ.

Given an operation $\mathcal{F}$ which takes a set and returns another set, we say that $\mathcal{F}$ is definable in families if for any family of definable sets $\{X_a\}_{a\in B}$ given by the formulae $\psi(x;y)$ and $\theta(y)$, there is formula $\phi(z;y)$ so that for each $a \in B$, the set $\mathcal{F}(X_a)$ is defined by $\phi(z;a)$.

Of particular importance for us will be several specific incarnations of $\mathcal{F}$ from the previous definition. Consider the definable family $\{X_a\}_{a\in B}$ (which we assume to be definable in some theory expanding the theory of fields). If $\mathcal{F}$ takes the set of points X_a and returns the Zariski closure of the set over K, then proving that $\mathcal{F}$ is definable in families in a given theory amounts to establishing a bound for the degree of the polynomials which define the Zariski closure of X_a which depends on the formula defining X_a, but is independent of the element a. In the theory of differentially closed fields, the main results of [9] establish results of this form.

Given a definable family $\{X_a\}_{a\in B}$ of Zariski closed sets over some field K, another important example occurs when $\mathcal{F}$ takes the closed set $X_a \subseteq \mathbb{A}^n$ and returns the set of components of X_a. In this case, we need to take some care in presenting $\mathcal{F}$ as an operation. Strictly speaking, $\mathcal{F}(X_a)$ is a finite set of algebraic subvarieties of $\mathbb{A}^n$. In general, if $\{X_a\}_{a\in B}$ is a constructible family of (possibly reducible) varieties, then there is a bound N on the number of irreducible components of X_a depending just on the family. Thus, $\mathcal{F}(X_a)$ may be presented as a sequence of subvarieties of $\mathbb{A}^n$ of length at most N, up to reordering. Using the theory of symmetric polynomials (or elimination of imaginaries relative to the theory of algebraically closed fields), such a finite set (or a finite sequence up to permutation) may be represented by a finite sequence.

We will require the following facts about definability in algebraically closed fields.

Fact 2.22. *We work relative to the theory of algebraically closed fields (ACF).*

1. *The Zariski closure is definable in families.*
2. *The dimension and degree of the Zariski closure of a set are definable in families.*
3. *Irreducibility of the Zariski closure is a definable property. More generally, the number of components of the Zariski closure is definable in families.*
4. *If the Zariski closure is an irreducible hypersurface given by the vanishing of some nonzero polynomial, then the degree of that polynomial in any particular variable is definable in families.*
5. *The family of irreducible components of the Zariski closure is definable in families.*

Fact 2.22 is established in the Appendix A.7. As we noted in the introduction, other proofs appear in the literature.

2.3.1. *Definability in the theory of differentially closed fields.* We now return to differential fields and develop results about definability with respect to the theory of differentially closed fields in this section. In particular, we will show how far we can go for the differential analogs of results in Fact 2.22. In the first place, we show that the differential dimension and order are definable in families.

In [12], intersections of differential varieties with generic differential hyperplanes were analyzed. The coefficients of the defining equation of a generic differential hyperplane, $u_0 + u_1x_1 + \cdots + u_nx_n = 0$, are taken to be differential indeterminates over the differential field K over which the variety is defined and various aspects of the geometry of the resulting intersection is established over the field $K\langle u_0,\dots,u_n\rangle$. In particular, the following result was proved.

Theorem 2.23. [8, 12] *Let $V\subseteq\mathbb{A}^n$ be an irreducible affine differential variety of differential dimension d and order h. Let H be a generic differential hyperplane defined by a linear form $u_0+u_1x_1+\cdots+u_nx_n$, whose coefficients u_i are differentially independent over K. Then over $K\langle u_0,\dots,u_n\rangle$, $V\cap H$ is nonempty if and only if $d>0$. In the case $d>0$, $V\cap H$ is an irreducible differential variety of differential dimension $d-1$ and order h.*

One can use Theorem 2.23 to prove the definability of dimension and order; as we have remarked above, there seem to be various other ways to prove these results.

Lemma 2.24. *Given a differentially constructible family of differential varieties $(X_s)_{s\in S(\mathfrak{U})}$, with $\dim(S)=0$, the set $\{s\in S\ :\ \dim(X_s)=d\}$ is a differentially constructible subset of S*

Proof. Fix $d+1$ tuples $(c_{i,j})_{1\le i\le d+1, 0\le j\le n}$ of length $n+1$ such that all these $c_{i,j}$ are differentially independent over K. Then by Theorem 2.23, for $s\in S$, $\dim(X_s)=d$ is equivalent to the condition

$$X_s\cap\mathbb{V}\big(\{c_{i,0}+\sum_{j=1}^n c_{i,j}y_j\}_{i=1}^d\big)\ne\emptyset \text{ and } X_s\cap\mathbb{V}\big(\{c_{i,0}+\sum_{j=1}^n c_{i,j}y_j\}_{i=1}^{d+1}\big)=\emptyset.$$

□

One should note that Theorem 2.23 applies in this case only because over any base of S, we know that any point on S is of differential transcendence degree 0. So, choosing some collection of independent differential transcendental elements over the base of all of the definable sets, the collection is independent and differentially transcendental over any given point in S.

Lemma 2.25. [1] *Differential dimension is definable in families. That is, given a differentially constructible family of differential varieties $(X_s)_{s\in S}$ and a number $d\in\mathbb{N}$ the set $\{s\in S:\ \dim(X_s)=d\}$ is a differentially constructible subset of S.*

Proof. Adopt the notation of Lemma 2.24. Suppose that $\dim(S)=n_1$. Then pick $2n_1+1$ systems of $d(n+1)$-tuples of mutually independent differential transcendentals (equivalently, in model theoretic terms, fix an indiscernible set in the generic type, over K; then pick any $(2n_1+1)d(n+1)$ elements from this set). Denote the chosen elements

$$\{c_{k,i,j}\ :\ 1\le k\le 2n_1+1,\ 1\le i\le d,\ 0\le j\le n\}$$

Of course, for any $s\in S$, over $\mathbb{Q}\langle s\rangle$, some of the $2n_1+1$ systems do not give generic independent sets of hyperplanes. But, because $\dim(S)=n_1$ and the systems are mutually independent, at least n_1+1 of the systems are generic over $\mathbb{Q}\langle s\rangle$ for any given $s\in S$.

Now, the requirement that $\dim(X_s)\ge d$ is equivalent to the condition that for at least n_1+1 values of the k,

$$X_s\cap\mathbb{V}(c_{k,1,0}+\sum_{j=1}^n c_{k,1,j}y_j,\dots,c_{k,d,0}+\sum_{j=1}^n c_{k,d,j}y_j)\ne\emptyset\ .$$

[1] The proof presented here is similar to the technique used in [7, Section 8.6].

□

Remark 2.26. Lemma 2.25 admits another proof. A differential constructible set $X \subseteq \mathbb{A}^n$ has differential dimension at least d just in case there is some coordinate projection $\pi : \mathbb{A}^n \to \mathbb{A}^d$ for which $\pi(X)$ is Kolchin dense in $\mathbb{A}^d$. In general, we do not know how to test definably whether some constructible set is Kolchin dense in another differential variety, but in the case of affine d-space, a constructible set is dense if and only if it is generic in Poizat's sense for the additive group structure, that is, finitely many additive translates of $\pi(X)$ cover $\mathbb{A}^d$. An easy Lascar rank computation shows that $d+1$ translates would suffice.

The order of a family of zero-dimensional differential varieties is definable in families [20, Appendix A.1]. The general result follows by reducing to this case via an argument similar to the proof of Lemma 2.25; See [7, section 8.6] for complete details.

Lemma 2.27. *The order of a definable set is definable in families.*

Proof. The proof is similar to the argument given in [7, Theorem 8.6.3]. If $(X_s)_{s\in S}$ is a family of differential varieties, and S has differential dimension n_1, then picking $2n_1+1$ many systems of $(d+1)(n+1)$-tuples of independent differential transcendentals $\{(a_{i,j}) \,|\, i = 1,\dots,2n_1+1,\ j = 1,\dots,(d+1)(n+1)\}$, then for any $s \in S$, at least n_1+1 of $\{(a_{k,j}) \,|\, j = 1,\dots,(d+1)(n+1)\}$ is a collection of independent differential transcendentals over s. Any such collection of tuples determins $d+1$-many independent generic hyperplanes in $\mathbb{A}^n$ over s. Thus by the main theorem of [8], the order of the intersection of X_s with this system of hyperplanes is equal to the order of X_s. So, the order of X_s is equal to the order given by the intersection of X_s with the $d+1$ many hyperplanes for at least n_1+1 many choices of the hyperplane system. So, we are reduced to showing that the order of a family of zero dimensional differential varieties is definable in families.

Given a zero-dimensional differential variety $X \subset \mathbb{A}^n$ defined by a collection of differential equations of order bounded by h, such that the zero set has differential dimension zero, we may write X as a D-variety in the sense of [20, Proposition 1.1] of $\mathbb{A}^{n(h+1)}$, and the order of X is given by the transcendence degree of the underlying algebraic variety. Algebraic dimension of algebraic varieties is definable in families, and thus so is the order. □

In section 2.2, prolongation admissible varieties are defined and important properties are developed. This special class of varieties plays an important role in Section 5. The next lemma shows that prolongation admissible is a definable condition, which follows from Fact 2.22 and the definition of prolongation admissible.

Lemma 2.28. *Prolongation admissibility is definable in families.*

Proof. Let $(V_b)_{b\in B}$ be a definable family of algebraic varieties in $\tau_h\mathbb{A}^n$ with V_b defined by $f_i(b, x, x', \dots, x^{(h)}) = 0$, $i = 1,\dots,m$, where $x^{(i)} = (x_1^{(i)},\dots,x_n^{(i)})$. By abuse of notation, let $B_h(V_b)$ be the Zariski closure of $\{\nabla_h(\bar{a}) : \nabla_h(\bar{a}) \in V_b\}$ in $\tau_h\mathbb{A}^n$. Then $\deg(B_h(V_b))$ has a uniform bound T in terms of the degree bound D of f_i, m, n and h. Indeed, let $z_{ij}\,(i = 1,\dots,n; j = 0,\dots,h)$ be new differential variables and replace $x_i^{(j)}$ by z_{ij} in each f_i to get a new differential polynomial g_i. Consider the new differential system $S := \{g_1,\dots,g_m, \delta(z_{i,j-1}) - z_{ij} : j = 1,\dots,h\}$. Regard S as a pure algebraic polynomial system in z_{ij} and $\delta(z_{ij})$ temporarily, and let U be the Zariski closed set defined by S in $\tau(\tau_h\mathbb{A}^n)$. Let $Z = \{\bar{c} = (c_{10}, \ldots, c_{n0}, \ldots, c_{1h}, \ldots, c_{nh}) \in \tau_h\mathbb{A}^n : (\bar{c}, \delta(\bar{c})) \in U\}$. Clearly, $Z = \{\nabla_h(\bar{a}) : \nabla_h(\bar{a}) \in V_b\}$. By [21, Remark 3.2], the degree of the Zariski closure of Z, namely $B_h(V_b)$, is bounded by $D_1 = D^{m(2^{n(h+1)}-1)}$.

By [16, Proposition 3], an irreducible algebraic variety V can be defined by $n(h+1)+1$ polynomials of degree bounded by the degree of V. So $B_h(V_b)$ can be defined by at most $(n(h+1)+1)^{D_1}$ polynomials of degree bounded by D_1^2. Hence, $(B_h(V_b))_{b\in B}$ is a definable family. Recall that V_b is prolongation admissible if and only if $V_b = B_h(V_b)$, which implies that $\{b : V_b \text{ is prolongation admissible}\}$ is a definable set. Thus, prolongation admissibility is definable in families. □

By Lemma 2.28 and Fact 2.22, we are safe to talk about definable families of irreducible prolongation admissible varieties. Given a definable family $\{X_a\}_{a\in B}$ of irreducible prolongation admissible varieties in $\tau_h\mathbb{A}^n$, as illustrated in Remark 2.16, the Kolchin polynomial of the dominant component of the differential variety corresponding to the prolongation sequence generated by X_a is determined by the dimensions of X_a and $\pi_{h,h-1}(X_a)$. So the order and the differential dimension of the dominant component of the corresponding differential variety are definable in families. This fact will be used in Section 5.

2.3.2. *The Ritt problem.* To establish the differential analog of Fact 2.22, it is natural to ask whether the Kolchin closure and irreducibility of *differential* varieties are definable in families. However, neither of these are known to be definable in families. This essentially comes down to the fact that it might not be possible to bound the orders of the differential polynomials which witness the non-primality of the differential ideal only from geometric datas. Developing such a bound is equivalent to several problems considered by Ritt [15, for instance, see the statement of Theorem 5.7 along with the references in the following remark], and we will refer to the development of such a bound as the *Ritt problem.*

Characteristic sets are *an* answer to this problem; various properties become definable in families of characteristic sets. The difficulties associated with the Ritt problem are the reason that our approach in this paper uses both prolongation sequences and characteristic sets.

The drawback of characteristic sets is that for points p such that the product of the separants of a given characteristic set vanish at p, determining if p is in the differential variety with a given characteristic set is an open problem [20, see the discussion in the appendices beginning on page 4286]. In this paper, we will parameterize characteristic sets of certain differential cycles rather than parameterizing generators of differential ideals. One might seek a more direct parameterization by generators of differential ideals, but doing so while following our general strategy would, at least on the surface, seem to require a solution to the Ritt problem.

Here is a specific indication of the problems that can arise when working directly with the generating sets of differential ideals; the following example shows that the order, h, will not suffice for the sort of bound described in the previous paragraphs.

Example 2.29. [20] Let $V = \mathbb{V}(2x^{(1)}x^{(3)} - (x^{(2)})^2 - 2x)$. Differentiating the defining equation results in the equation $2x^{(1)}(x^{(4)} - 1) = 0$. From this, it is easy to see that V consists of two components, $x = 0$ and the generic component.

Of course, more differentiations might be necessary:

Example 2.30. [35] Consider $V = \mathbb{V}(f)$ where $f = (y^{(2)})^2 - y \in K\{y\}$. Differentiating f successively 3 times, one obtains

$$\begin{aligned} \delta f &= 2y^{(2)}y^{(3)} - y^{(1)} \\ \delta^2 f &= 2y^{(2)}y^{(4)} + 2(y^{(3)})^2 - y^{(2)} \\ \delta^3 f &= 2y^{(2)}y^{(5)} + 6y^{(3)}y^{(4)} - y^{(3)} \end{aligned}$$

Then $2y^{(3)} \cdot \delta^3 f - (6y^{(4)} - 1)f^{(2)} = y^{(2)}(4y^{(3)}y^{(5)} - 12(y^{(4)})^2 + 8y^{(4)} - 1) \in [f]$. Thus, $V = \mathbb{V}(f, y^{(2)}) \cup \mathbb{V}(f, 4y^{(3)}y^{(5)} - 12(y^{(4)})^2 + 8y^{(4)} - 1)$ is reducible.

Informally, the Ritt problem asks if there is an upper bound to the number of required differentiations in terms of the "shape" of the equations. An equivalent form of the Ritt problem [20, Appendix 1] is testing when a given point (say 0) at which the separants of the characteristic set vanish is in the generic component of the differential ideal generated by the characteristic set.

Remark 2.31. Note that although many of the arguments and bounds in this paper are theoretical, most of them could be made effective (for instance, many of the definability arguments could be made effective using differential elimination algorithms).

3. Algebraic Chow forms and differential Chow forms

In this section we recall the definitions of Chow forms, Chow varieties, and their differential algebraic analogs. The algebraic Chow form was first defined for projective varieties by Chow [3]. When $\sum_{i=0}^n c_i y_i$ is a linear form in $y_0, \ldots, y_n$ with coefficients $\{c_i\}_{i=0}^n$ a tuple of independent transcendentals, we call the form algebraically generic, and we call the zero set of such a form a generic hyperplane.

Definition 3.1. [3, 18] Let $V \subseteq \mathbb{P}^n$ be an irreducible projective variety of dimension d. Take d independent generic linear forms $L_i = v_{i0}y_0 + \cdots + v_{in}y_n$ for $1 \leq i \leq d$), then V intersects $V(L_1, \ldots, L_d)$ in a finite set of points, say $(\xi_{\tau 0}, \ldots, \xi_{\tau n})\,(\tau = 1, \ldots, m)$. Then there exists a polynomial $A \in K[\mathbf{v}_1, \ldots, \mathbf{v}_d]$ such that $F(\mathbf{v}_0, \ldots, \mathbf{v}_d) = A \prod_{\tau=1}^m (\sum_{j=0}^n v_{0j}\xi_{\tau j})$ is an irreducible polynomial in $K[\mathbf{v}_0, \ldots, \mathbf{v}_d]$ where $\mathbf{v}_i = (v_{i0}, v_{i1}, \ldots, v_{in})$. This F is called the *algebraic Chow form* of V.

The Chow form F is homogeneous in each $\mathbf{v}_i$ of degree m. We call m the *degree* of V, denoted by $\deg(V)$. Throughout the remainder of this paper, unless otherwise indicated, varieties and differential varieties are affine. We now introduce the concept of algebraic Chow form for irreducible varieties in $\mathbb{A}^n$.

Definition 3.2. Let $V \subseteq \mathbb{A}^n$ be an irreducible affine variety of dimension d. Let $V' \subseteq \mathbb{P}^n$ be the projective closure of V with respect to the usual inclusion of $\mathbb{A}^n$ in $\mathbb{P}^n$ (identifying $(a_1, \ldots, a_n) \in \mathbb{A}^n$ with $[1 : a_1 : \ldots : a_n] \in \mathbb{P}^n$). We define the algebraic Chow form of V to be the algebraic Chow form of V'.

An (effective) *algebraic cycle* in $\mathbb{A}^n$ of dimension d over K is of the form $V = \sum_{i=1}^\ell t_i V_i\,(t_i \in \mathbb{Z}_{\geq 0})$ where each V_i is an irreducible variety of dimension d in $\mathbb{A}^n$. We define the algebraic Chow form of V to be $F(\mathbf{v}_0, \ldots, \mathbf{v}_d) = \prod_{i=1}^\ell (F_i(\mathbf{v}_0, \ldots, \mathbf{v}_d))^{t_i}$ where F_i is the algebraic Chow form of V_i, and define the degree of V to be $\sum_{i=1}^\ell t_i \deg(V_i)$, which is the homogenous degree of F in each $\mathbf{v}_i$. The coefficient vector of F, regarded as a point in a projective space, is correspondingly called the

Chow coordinate of V. Each algebraic cycle is uniquely determined by its algebraic Chow form, in other words, determined by its Chow coordinate.

In [3], Chow proved that the set of all algebraic cycles in $\mathbb{P}^n$ of dimension d and degree m in the Chow coordinate space is a projective variety, now called the Chow variety of index (d, m). In general, the set of all algebraic cycles in $\mathbb{A}^n$ of dimension d and degree m is not closed in the Chow coordinate space. Below, we give a simple example.

Example 3.3. Consider the set X of all algebraic cycles in $\mathbb{A}^2$ of dimension 0 and degree 1. Each $V \in X$ can be represented by two linear equations $a_{i0} + a_{i1}y_1 + a_{i2}y_2 = 0\,(i = 0, 1)$ with $a_{01}a_{12} - a_{02}a_{11} \neq 0$. Then the Chow form of V is $F(v_{00}, v_{01}, v_{02}) = (a_{01}a_{12} - a_{02}a_{11})v_{00} - (a_{00}a_{12} - a_{02}a_{10})v_{01} + (a_{00}a_{11} - a_{01}a_{10})v_{02}$. So the Chow coordinate of V is $(a_{01}a_{12} - a_{02}a_{11}, -a_{00}a_{12} + a_{02}a_{10}, a_{00}a_{11} - a_{01}a_{10})$. Thus, the Chow coordinates of cycles in X is the set $\{(c_0, c_1, c_2) : c_0 \neq 0\} = \mathbb{P}^2 \setminus V(c_0)$, which is not a closed variety, but is a constructible set.

The following result shows that the set of all cycles with given degree and dimension is always a constructible set in the Chow coordinate space.

Proposition 3.4. *The set of all algebraic cycles in $\mathbb{A}^n$ of dimension d and degree m is a constructible set in a higher dimensional projective space. We call this set the* affine Chow variety *of index (d, m) in $\mathbb{A}^n$, denoted by* $\mathrm{Chow}_n(d, m)$*, or* $\mathrm{Chow}(d, m)$ *if the space $\mathbb{A}^n$ is clear from the context.*

Proof. Let M be the set of all monomials in $\mathbf{v}_0, \ldots, \mathbf{v}_d$ which are of degree m in each $\mathbf{v}_i$. That is, $M = \{\prod_{i=0}^{d} \prod_{j=0}^{n} v_{ij}^{\sigma_{ij}} | \sigma_{ij} \in \mathbb{Z}_{\geq 0}, \sum_{j=0}^{n} \sigma_{ij} = m\}$. Let $F_0 = \sum_{\phi \in M} c_\phi \phi$ where c_ϕ are algebraic indeterminates over K. By [3, 18], there exists a projective variety $W \subseteq \mathbb{P}^{|M|-1}$ such that $(\bar{c}_\phi : \phi \in M) \in W$ if and if $\bar{F}_0 = \sum_{\phi \in M} \bar{c}_\phi \phi$ is the algebraic Chow form of an algebraic cycle in $\mathbb{P}^n$ of dimension d and degree m.

Let $N = \{v_{00}^{m} \prod\limits_{i=1}^{d} \prod\limits_{j=0}^{n} v_{ij}^{\sigma_{ij}} | \sigma_{ij} \in \mathbb{Z}_{\geq 0}, \sum_{j=0}^{n} \sigma_{ij} = m\} \subseteq M$ and let $\{c_1, \ldots, c_{|N|}\}$ be the set of all coefficients of F_0 with respect to monomials contained in N. Let $W_1 = W \setminus V(c_1, \ldots, c_{|N|})$, where $V(c_1, \ldots, c_{|N|}) \subseteq \mathbb{P}^{|M|-1}$ temporarily denotes the projective variety defined by $c_1 = \cdots = c_{|N|} = 0$. We claim that there is a one-to-one correspondence between $\mathrm{Chow}_n(d, m)$ and W_1 via algebraic Chow forms. On the one hand, for each point in $\mathrm{Chow}_n(d, m)$ corresponding to an algebraic cycle V, the algebraic Chow form $F = \sum_{\phi \in M} \bar{c}_\phi \phi$ of V has the following Poisson-type product formula: $F = A \prod_{\tau=1}^{m} (v_{00} + \sum_{j=1}^{n} v_{0j}\xi_{\tau j})$ where $A \in k[\mathbf{v}_1, \ldots, \mathbf{v}_d]$ and $(\xi_{\tau 1}, \ldots, \xi_{\tau n})$ is a generic point of a component of V. Thus, there exists at least one monomial $\phi \in N$ such that ϕ appears effectively in F. As a consequence, $(\bar{c}_\phi) \in W_1$. On the other hand, for each $(\bar{c}_\phi : \phi \in M) \in W_1$, $\bar{F}_0 = \sum_{\phi \in M} \bar{c}_\phi \phi$ is the algebraic Chow form of an algebraic cycle $\bar{V}' = \sum_i t_i V_i'$ in $\mathbb{P}^n$ of dimension d and degree m. Also from the Poisson-product formula, we can see that each $V_i' \not\subseteq \mathbb{U}_0$, where $\mathbb{U}_0$ is the particular open set of $\mathbb{P}^n(K)$ determined by $y_0 \neq 0$. Suppose $(1, a_{i1}, \ldots, a_{in}) \in \mathbb{P}^n$ is a generic point of V_i'. Let $V_i \subset \mathbb{A}^n$ be the affine variety with $(a_{i1}, \ldots, a_{in})$ as a generic point. Thus, $\bar{F}_0$ is the algebraic Chow form of the algebraic cycle $\bar{V} = \sum_i t_i V_i \in \mathrm{Chow}_n(d, m)$. Hence, we have proved that $\mathrm{Chow}_n(d, m)$ is a constructible set. □

Algebraic Chow forms can uniquely determine the corresponding algebraic varieties. In particular, the defining equations of an irreducible variety can be recovered from its Chow form; see [18, p. 51] or [12, Theorem 4.45]. Thus, we can obtain the following lemma which will be needed in Section 5.

Lemma 3.5. *The set of irreducible varieties in $\mathbb{A}^n$ of dimension d and degree m is a definable family.*

Proof. Let $C \subseteq \text{Chow}_n(d,m)$ be the set of Chow coordinates of all irreducible varieties of dimension d and degree m in $\mathbb{A}^n$. Then C is a constructible set. Indeed, given $c \in \text{Chow}_n(d,m)$, c is the Chow coordinate of an irreducible variety if and only if the corresponding polynomial $\bar{F}_0$ with coefficient vector c defined as in the proof of Proposition 3.4 is irreducible, which is a definable condition. Given $c \in C$, the corresponding polynomial $\bar{F}_0$ is the Chow form of some irreducible variety V_c. By the algebraic analog of [12, Theorem 4.45], two subsets of polynomials in $x_1, \ldots, x_n$ with coefficients linear in c, say S_1 and S_2, can be computed from $\bar{F}_0$, such that V_c is the Zariski closure of the quasi-variety $V(S_1)\backslash V(S_2)$. Since Zariski closure is definable in families with respect to ACF, $(V_c)_{c\in C}$, the set of irreducible varieties in $\mathbb{A}^n$ of dimension d and degree m, is a definable family. □

In the remainder of this section, we recall the definitions and properties of differential Chow forms and propose the main problem we are considering in this paper. Let $V \subseteq \mathbb{A}^n$ be an irreducible differential variety defined over K of dimension d and

$$L_i = u_{i0} + u_{i1}y_1 + \cdots + u_{in}y_n \; (i = 0, \ldots, d)$$

be $d+1$ differentially generic inhomogeneous linear forms. For each i, denote $\mathbf{u}_i = (u_{i0}, u_{i1}, \ldots, u_{in})$. Let

$$\mathcal{I}_{\mathbf{u}} = [\mathbb{I}(V), L_0, \ldots, L_d]_{K\{y_1,\ldots,y_n,\mathbf{u}_0,\ldots,\mathbf{u}_d\}} \cap K\{\mathbf{u}_0, \ldots, \mathbf{u}_d\}. \tag{2}$$

Then by [12, Lemma 4.1], $\mathcal{I}_{\mathbf{u}}$ is a prime differential ideal in $K\{\mathbf{u}_0, \ldots, \mathbf{u}_d\}$ of codimension one.

Definition 3.6. The *differential Chow form* of V or $\mathbb{I}(V)$ is defined as the unique (up to appropriate scaling) irreducible differential polynomial $F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ such that $\mathcal{I}_{\mathbf{u}} = \text{sat}(F)$ under any differential ranking.

Note that from Definition 3.6, the differential Chow form F itself is a characteristic set of $\mathcal{J}_{\mathbf{u}}$ under any differential ranking. Differential Chow forms uniquely characterize their corresponding differential ideals. The following theorem gives some basic properties of differential Chow forms.

Theorem 3.7. [12] *Let V be an irreducible differential variety defined over K with differential dimension d and order h. Suppose $F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ is the differential Chow form of V. Then F has the following properties.*

1) $\text{ord}(F) = h$. *In particular,* $\text{ord}(F, u_{i0}) = h$ *for each* $i = 0, \ldots, d$.
2) *F is differentially homogenous of the same degree m in each $\mathbf{u}_i$. This m is called the* differential degree *of V.*
3) *Let $g = \deg(F, u_{00}^{(h)})$. There exist elements $\xi_{\tau j} \in \mathcal{U}$ for $\tau = 1, \ldots, g$ and $j = 1, \ldots, n$ such that*

$$F = A \prod_{\tau=1}^{g} (u_{00} + u_{01}\xi_{\tau 1} + \cdots + u_{0n}\xi_{\tau n})^{(h)}$$

where A is a differential polynomial free from $u_{00}^{(h)}$. Moreover, each $\xi_\tau = (\xi_{\tau 1}, \ldots, \xi_{\tau n})$ is a generic point of V and $L_1, \ldots, L_d$ all vanish at ξ_τ.

4) *The algebraic variety* $B_h(V) \cap V(L_1^{(h)}, \cdots, L_d^{(h)}, L_0^{(h-1)}) \subseteq \tau_h \mathbb{A}^n$ *is of dimension zero. Its cardinality,* g*, is called the* leading differential degree *of* V*. Here, the* $L_i^{(j)}$ *are considered as polynomials in variables* $y_1^{[h]}, \ldots, y_n^{[h]}$*.*

We give a simple example to illustrate these invariants of a differential variety.

Example 3.8. Let $n = 1$ and $V = \mathbb{V}(y^2 y' + 1) \subseteq \mathbb{A}^1$. Then the differential Chow form of V is $F(\mathbf{u}_0) = u_{00}^2 u_{01} u_{00}' - u_{00}^3 u_{01}' - u_{01}^4$. The order of V is 1, the differential degree of V is 4 and the leading differential degree of V is 1.

The following lemma is useful in the computation of differential Chow forms from the point view of algebraic ideals, which follows directly from the definition and properties of the differential Chow form.

Lemma 3.9. *Let* V *be an irreducible differential variety defined over* K *with differential dimension* d *and order* h*. Let* $F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ *be the differential Chow form of* V*. Then*

$$(F) = \big(I(B_h(V)), L_0^{[h]}, \ldots, L_d^{[h]}\big) \cap K[\boldsymbol{u}_0^{[h]}, \ldots, \boldsymbol{u}_d^{[h]}]. \tag{3}$$

Proof. By the definition of differential Chow form, $\mathcal{I}_{\mathbf{u}} = \mathrm{sat}(F)$ under any ranking. Since $\mathrm{ord}(F) = h$ by Theorem 3.7, $\mathcal{I}_{\mathbf{u}} \cap K[\mathbf{u}_0^{[h]}, \ldots, \mathbf{u}_d^{[h]}] = (F)$. Let $\mathcal{J} = \big(I(B_h(V)), L_0^{[h]}, \ldots, L_d^{[h]}\big) \cap K[\mathbf{u}_0^{[h]}, \ldots, \mathbf{u}_d^{[h]}]$. Take a generic point $\xi = (\xi_1, \ldots, \xi_n)$ of $\mathbb{I}(V)$ such that the u_{ij} are differentially independent over $K\langle\xi\rangle$. Let $\zeta_i = -\sum_{j=1}^n u_{ij}\xi_j$. It is easy to show that $(\zeta_0^{[h]}, \ldots, \zeta_d^{[h]})$ is a generic point of both $\mathcal{J}$ and $\mathcal{I}_{\mathbf{u}} \cap K[\mathbf{u}_0^{[h]}, \ldots, \mathbf{u}_d^{[h]}]$, so the two ideals are equal, which implies (3). □

A differential variety is called *order-unmixed* if all its components have the same differential dimension and order. Let V be an order-unmixed differential variety of dimension d and order h and $V = \bigcup_{i=1}^{\ell} V_i$ its minimal irreducible decomposition with $F_i(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)$ the differential Chow form of V_i. Let

$$F(\mathbf{u}_0, \ldots, \mathbf{u}_d) = \prod_{i=1}^{\ell} F_i(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_d)^{s_i} \tag{4}$$

with s_i arbitrary nonnegative integers. In [12], a *differential algebraic cycle* is defined associated to (4) similar to its algebraic analog, that is, $\mathbf{V} = \sum_{i=1}^{\ell} s_i V_i$ is a differential algebraic cycle with s_i as the multiplicity of V_i and $F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ is called the differential Chow form of $\mathbf{V}$.

Suppose each V_i is of differential degree m_i and leading differential degree g_i, then the leading differential degree and differential degree of $\mathbf{V}$ is defined to be $\sum_{i=1}^{\ell} s_i g_i$ and $\sum_{i=1}^{\ell} s_i m_i$ respectively.

Definition 3.10. A differential cycle $\mathbf{V}$ in the n dimensional affine space $\mathbb{A}^n$ with dimension d, order h, leading differential degree g, and differential degree m is said to be of index (d, h, g, m) in $\mathbb{A}^n$.

Definition 3.11. Let $\mathbf{V}$ be a differential cycle of index (d, h, g, m) in $\mathbb{A}^n$. The *differential Chow coordinate* of $\mathbf{V}$ is the coefficient vector of the differential Chow form of $\mathbf{V}$ considered as a point in a higher dimensional projective space determined by (d, h, g, m) and n.

Definition 3.12. Fix an index (d, h, g, m) and n. To each differential field k we associate the set

$$\mathbb{C}_{(n,d,h,g,m)}(k) := \{\mathbf{V} : \mathbf{V} \text{ is a differential cycle of index } (d,h,g,m) \text{ in } \mathbb{A}^n \text{ over } k\}$$

thereby defining a functor from the category of differential fields to the category of sets. If this functor is represented by some differentially constructible set, meaning that there is a differentially constructible set and a natural isomorphism between the functor $\mathbb{C}(n, d, h, g, m)$ and the functor given by this differentially constructible set (regarded also as a functor from the category of differential fields to the category of sets), then we call this differentially constructible set the *differential Chow variety* of index (d, h, g, m) of $\mathbb{A}^n$ and denote it by δ-Chow(n, d, h, g, m). In this case, we also say the differential Chow variety δ-Chow(n, d, h, g, m) exists.

Theorem 3.13. [12, Theorem 5.7] *In the case $g = 1$, the differential Chow variety δ-Chow$(n, d, h, 1, m)$ exists.*

The above theorem was proved with constructive methods. But that method does not apply to the general case (when g is arbitrary) and the existence of differential Chow varieties was listed as an open problem in [12]. In section 5, we will give a positive answer to the very problem using model theoretical methods, namely, showing that the differential Chow varieties exist for general cases.

Remark 3.14. In the algebraic setting, for an arbitrary tuple (d, m), Chow$_n(d, m)$ is always a nonempty constructible set. However, it is more subtle in the differential case and $\mathbb{C}_{(n,d,h,g,m)}$ may be empty for certain values (n, d, h, g, m). For example, when a differential algebraic cycle is of order 1, its differential degree is at least 2, so $\mathbb{C}_{(n,d,1,g,1)} = \emptyset$.

4. Degree bound for prolongation sequences

We are interested in the space of all differential cycles in n dimensional affine space of some fixed index (d, h, g, m). Ultimately, the point in our parameter space corresponding to a differential cycle $\sum_i a_i V_i$ will be given by the point representing $\sum_i a_i B_h(V_i)$ in an appropriate algebraic Chow variety. In order to ensure that the space of such algebraic varieties has the structure of a definable set, we must establish degree bounds for the corresponding algebraic cycles. This is the topic of the present section.

Proposition 4.1. *Suppose V is an irreducible differential variety of index (d, h, g, m) in $\mathbb{A}^n$. Then there is a natural number D depending only on (d, h, g, m) such that $B_h(V) \subseteq \tau_h\mathbb{A}^n$ is an irreducible algebraic variety with degree satisfying* $\deg(B_h(V)) \leq D$.

Proof. The irreducibility of $B_h(V)$ follows from the fact that $B_h(V) = V\big(\mathbb{I}(V) \cap K\{x_1, \ldots, x_n\}_{\leq h}\big)$. It remains for us to show that there is D with the claimed properties.

Suppose $F(\mathbf{u}_0, \ldots, \mathbf{u}_d)$ is the differential Chow form of V where $\mathbf{u}_i = (u_{i0}, \ldots, u_{in})$ $(i = 0, \ldots, d)$. Let $\mathbf{u}$ be the tuple of variables $(u_{ij})_{i=0,j=1}^{d,n}$. That is, we are omitting the variables of the form u_{i0}. Set $K_1 = K\langle\mathbf{u}\rangle$. Let W be the differential variety in $\mathbb{A}^{d+1}$ defined by sat(F) considered as a differential ideal in $K_1\{u_{00}, \ldots, u_{d0}\}$. Then by Theorem 3.7, $B_h(W) = V(F) \subseteq \tau_h\mathbb{A}^{d+1}$ is an irreducible variety.

By [12, Theorem 4.13], the map given by

$$f(\mathbf{u}_0) = (\frac{\partial F}{\partial u_{0i}^{(h)}}/S_F)_{i=1}^n$$

gives a differential birational map from W to V_{K_1}, the base change of V to K_1. By quantifier elimination in DCF_0, the image is given by the vanishing and non vanishing of some collection of differential polynomials. By the compactness theorem, the number, degree and order of these equations and inequations must be bounded uniformly depending only on the degrees, orders, and number of variables of F and f. The results of [21] (see Remark 3.2) give a uniform upper bound, D for the degree of $B_h(V)$. □

Corollary 4.2. *Suppose* $\mathbf{V} = \sum_i a_i V_i \subseteq \mathbb{A}^n$ $(a_i \in \mathbb{Z}_{\geq 0})$ *is an order-unmixed differential variety of index* (d,h,g,m)*. Then there is a natural number* D *such that* $\sum_i a_i B_h(V_i)$ *is an algebraic cycle in* $\tau_h\mathbb{A}^n$ *of dimension* $d(h+1)+h$ *and degree satisfying* $\deg(B_h(V)) \leq D$

It is possible to give effective versions of Proposition 4.1 and Corollary 4.2 with a more complicated proof; the following proposition gives such detailed effective bounds. In the following section of the paper, we will use Corollary 4.2 to restrict the space of algebraic Chow varieties which we consider. A more detailed analysis of the particular defining equations of the differential Chow variety might be undertaken by applying the more detailed effective bounds of the following Proposition (or improving upon them), but the main thrust of our results in the next section concerns the *existence* of differential Chow varieties, so the following result is primarily given to indicate that the construction of differential Chow varieties can be made effective *in principle.*

Proposition 4.3. *Suppose* V *is an irreducible differential variety of index* (d,h,g,m) *in* $\mathbb{A}^n$*. Then* $B_h(V) \subseteq \tau_h\mathbb{A}^n$ *is an irreducible algebraic variety with degree satisfying*

$$\max\{g, m/(h+1)\} \leq \deg(B_h(V)) \leq [(d+1)m]^{nh+n+1}. \tag{5}$$

Proof. Suppose $F(\mathbf{u}_0,\ldots,\mathbf{u}_d)$ is the differential Chow form of V where $\mathbf{u}_i = (u_{i0},\ldots, u_{in})$ $(i=0,\ldots,d)$. Let $\mathbf{u} = (u_{ij})_{i=0,j=1}^{d,n}$ and $K_1 = K\langle\mathbf{u}\rangle$.

Let $\mathcal{J} = [\mathbb{I}(V), L_0,\ldots,L_d] \subseteq K_1\{x_1,\ldots,x_n,u_{00},\ldots,u_{d0}\}$. Then by the proof of [12, Theorem 4.36], the polynomials $g_{jk} = \frac{\partial F}{\partial u_{00}^{(h)}}x_j^{(k)} + \sum_{\ell=1}^k \binom{h-\ell}{k-\ell}/\binom{h}{k}\frac{\partial F}{\partial u_{00}^{(h-\ell)}}x_j^{(k-\ell)} - \frac{\partial F}{\partial u_{0j}^{(h-k)}}$ $(j=1,\ldots,n; k=0,\ldots,h)$ are contained in $\mathcal{J}$. Fix an ordering of algebraic indeterminates so that $x_1 < \cdots < x_n < x_1^{(1)} < \cdots < x_n^{(1)} < \cdots < x_1^{(h)} < \ldots < x_n^{(h)}$ and $u_{ij}^{(k)} < x_\ell^{(m)}$ for all i,j,k,ℓ, and m.

Let $\mathcal{J}^{[h]} := \mathcal{J} \cap K_1\{x_1,\ldots,x_n,u_{00},\ldots,u_{d0}\}_{\leq h}$. Since for each $f \in \mathcal{J}^{[h]}$, the algebraic remainder of f with respect to g_{jk} is a polynomial in $\mathcal{J}\cap K_1\{u_{00},\ldots,u_{d0}\}_{\leq h} = (F)$, $\{F\} \cup \{g_{jk} \,:\, 1 \leq j \leq n, 0 \leq k \leq h\}$ constitutes an algebraic characteristic set of $\mathcal{J}^{[h]}$. Thus,

$$\mathcal{J}^{[h]} = \big(F, (g_{jk})_{1\leq j\leq n; 0\leq k\leq h}\big) : \big(\frac{\partial F}{\partial u_{00}^{(h)}}\big)^\infty.$$

Since the variety defined by the ideal $\big(F, (g_{jk})_{1\leq j\leq n;0\leq k\leq h}\big) : \big(\frac{\partial F}{\partial u_{00}^{(h)}}\big)^\infty$ is a component of the closed set given by the vanishing of $\big(F, (g_{jk})_{1\leq j\leq n;0\leq k\leq h}\big)$, by [16, Theorem 1],

$$\begin{aligned}\deg(\mathcal{J}^{[h]}) &\leq \deg\left(\left(F,(g_{jk})_{1\leq j\leq n;0\leq k\leq h}\right)\right)\\ &\leq \deg(F)^{n(h+1)+1}\leq [(d+1)m]^{nh+n+1}.\end{aligned}$$

Since $\mathcal{J}^{[h]}\cap K_1\{x_1,\ldots,x_n\}_{\leq h}=I(B_h(V)_{K_1})$, by [16, 27], $\deg(B_h(V))\leq\deg(\mathcal{J}^{[h]})$. Hence, $\deg(B_h(V))\leq[(d+1)m]^{nh+n+1}$.

Since $\dim(B_h(V))=d(h+1)+h$ and by [12], $B_h(V)$ and some $d(h+1)+h$ hyperplanes defined by $L_0^{(i)}$ for $0\leq i<h$ and $L_1^{(i)},\ldots,L_d^{(i)}$ for $0\leq i\leq h$ intersect in g points, $\deg(B_h(V))\geq g$. On the other hand, F can be obtained from the algebraic Chow form of $B_h(V)$ using the strategy of specializations in [27, Theorem 4.2]. So $\deg(F)\leq(h+1)(d+1)\deg(B_h(V))$ and $\deg(F)=m(d+1)$. Thus, (5) follows. □

5. On the existence of differential Chow varieties

In this section, we will show that for a fixed $n\in\mathbb{N}$ and a fixed index (d,h,g,m), $\mathbb{C}_{(n,d,h,g,m)}$ is represented by a differentially constructible set. That is, the differential Chow variety δ-Chow(n,d,h,g,m) exists.

Consider the disjoint union of algebraic constructible sets

$$\mathcal{C}=\bigcup_{e\leq D}\mathrm{Chow}_{n(h+1)}(d(h+1)+h,e)$$

where D is the bound of Corollary 4.2 and $\mathrm{Chow}_{n(h+1)}(d(h+1)+h,e)$ is the affine algebraic Chow variety of index $(d(h+1)+h,e)$ in $\tau_h\mathbb{A}^n$ as defined in Proposition 3.4. So each point $a\in\mathcal{C}$ represents an algebraic cycle of the form $\sum_i t_iW_i$, where each W_i is an irreducible variety in $\tau_h\mathbb{A}^n$. Moreover, for a fixed i, if W_i is prolongation admissible, by Lemma 2.15, the differential variety corresponding to the prolongation sequence generated by W_i has a unique dominant component $V_i\subseteq\mathbb{A}^n$.

Let $\mathcal{C}_1$ be the subset consisting of all points $a\in\mathcal{C}$ such that

(1) a is the Chow coordinate of an algebraic cycle $\sum_i t_iW_i$ where each W_i is irreducible and prolongation admissible and
(2) for each i, the unique dominant component of the differential variety corresponding to the prolongation sequence generated by W_i is of index (d,h,g_i,m_i) and $\sum_i t_ig_i=g,\sum_i t_im_i=m$.

Theorem 5.1. *The set $\mathcal{C}_1$ is differentially constructible and the map which associates a differential algebraic cycle $\mathbf{V}=\sum s_iV_i$ of index (d,h,g,m) in $\mathbb{A}^n$ with the Chow coordinate of the algebraic cycle $\sum s_iB_h(V_i)$ identifies $\mathbb{C}_{(n,d,h,g,m)}$ with $\mathcal{C}_1$. In particular, the differential Chow variety δ-*Chow*(n,d,h,g,m) exists.*

Proof. First, we show $\mathcal{C}_1$ is a differentially constructible set. From the definition of Chow coordinates, we know each $\mathrm{Chow}_{n(h+1)}(d(h+1)+h,e)$ actually represents a definable family $S_e:=(F_c)_{c\in\mathrm{Chow}_{n(h+1)}(d(h+1)+h,e)}$ of homogenous polynomials which are Chow forms of algebraic cycles in $\mathbb{A}^{n(h+1)}$ of dimension $d(h+1)+h$ and degree e. Recall that the Chow coordinate c of a cycle is just the coefficient vector of the Chow form F_c of this cycle. By item 5) of Fact 2.22, the family of irreducible components of the definable family S_e is definable in families. Take an arbitrary $c\in\mathrm{Chow}_{n(h+1)}(d(h+1)+h,e)$ and the corresponding polynomial $F_c\in S_e$ for an example. Suppose F_c has the irreducible decomposition $F_c=$

$\prod_{i=1}^{\ell} F_{c,i}^{t_i}$, then each $F_{c,i}$ is the Chow form of an irreducible variety $W_{c,i}$ and F_c is the Chow form of the algebraic cycle $\sum_{i=1}^{\ell} t_i W_{c,i}$. To show $\mathcal{C}_1$ is differentially constructible, we need to consider each irreducible component $F_{c,i}$ in the above. By Lemma 3.5, the family of irreducible varieties $W_{c,i}$ is a definable family. And by Lemma 2.28, the family of irreducible and prolongation admissible varieties $W_{c,i}$ is a definable family. Let $V_{c,i}$ be the unique dominant component of the differential variety corresponding to the prolongation sequence generated by $W_{c,i}$. Then by Lemma 2.15 and Remark 2.16, the differential dimension of $V_{c,i}$ is equal to $d_1 = \dim(W_{c,i}) - \dim\left(\pi_{h,h-1}(W_{c,i})\right)$ and the order of V_{c_i} is equal to $\dim(W_{c,i}) - d_1(h+1)$. Since algebraic dimension is a definable property, the set of Chow coordinates of the algebraic cycles, each of whose irreducible component is prolongation admissible and generates a prolongation sequence such that the unique dominant component of the differential variety corresponding to this sequence is of differential dimension d and order h, is a definable set.

Suppose δ-$\dim(V_{c,i}) = d$ and $\mathrm{ord}(V_{c,i}) = h$. Let U be the algebraic variety in $\mathbb{A}^n \times \left(\mathbb{P}^{(n+1)(h+1)-1}\right)^{d+1}$ defined by the defining formulae of $W_{c,i}$ and $L_0^{[h]} = 0, \ldots, L_d^{[h]} = 0$ with each $L_i^{(j)} = u_{i0}^{(j)} + \sum_{k=1}^{n} \sum_{\ell=0}^{j} \binom{j}{\ell} u_{ik}^{(\ell)} x_k^{(j-\ell)}$ regarded as a polynomial in variables $x_k^{(j)}$ and $u_{ik}^{(\ell)}$. Since $B_h(V_{c,i}) = W_{c,i}$, by Lemma 3.9, the Zariski closure of the image of U under the following projection map

$$\pi : \mathbb{A}^n \times \left(\mathbb{P}^{(n+1)(h+1)-1}\right)^{d+1} \longrightarrow \left(\mathbb{P}^{(n+1)(h+1)-1}\right)^{d+1}$$

is an irreducible variety of codimension 1, and the defining polynomial F of $\overline{\pi(U)}$ is the differential Chow form of $V_{c,i}$. By item 4) of Fact 2.22, the total degree of F and $\deg(F, u_{00}^{(h)})$ are both definable in families; these quantities are the differential degree and the leading differential degree of $V_{c,i}$, respectively. So the differential degree and the leading differential degree of $V_{c,i}$ are definable in families. Hence, $\mathcal{C}_1$ is a definable set, and also a differentially constructible set due to the fact that the theory DCF_0 eliminates quantifiers.

By Lemma 2.15 and Remark 2.16, each algebraic cycle $\sum s_i W_i$ corresponding to a point of $\mathcal{C}_1$ determines a differential algebraic cycle $\sum s_i V_i \in \mathbb{C}(n,d,h,g,m)(\mathcal{U})$, where V_i is the unique dominant component of the differential variety corresponding to the prolongation sequence generated by W_i. And on the other hand, each differential algebraic cycle $\sum s_i V_i \in \mathbb{C}(n,d,h,g,m)(\mathcal{U})$ determines the corresponding algebraic cycle $\sum s_i B_h(V_i)$, which is an algebraic cycle corresponding to a point of $\mathcal{C}_1$, by Corollary 4.2. So we have established a natural one-to-one correspondence between $\mathbb{C}(n,d,h,g,m)(\mathcal{U})$ and $\mathcal{C}_1$. Thus, $\mathbb{C}(n,d,h,g,m)$ is represented by the differentially constructible set δ-$\mathrm{Chow}(n,d,h,g,m) := \mathcal{C}_1$. □

Remark 5.2. For the special case $d = n-1$, the existence of the differential Chow variety of index $(n-1,h,g,m)$ can be easily shown from the point of view of differential characteristic sets. Indeed, note that each order-unmixed radical differential ideal $\mathcal{I}$ of dimension $n-1$ and order h has the prime decomposition $\mathcal{I} = \bigcap_{i=1}^{t} \mathrm{sat}(f_i) = \mathrm{sat}(\prod_{i=1}^{t} f_i)$, where $f_i \in K\{x_1,\ldots,x_n\}$ is irreducible and of order h. Thus, there is a one-to-one correspondence between $\mathbb{C}_{(n,n-1,h,g,m)}(K)$ and the set of all differential polynomials $f \in K\{x_1,\ldots,x_n\}$ such that each irreducible component of f is of order h, $\deg(f, \{x_1^{(h)},\ldots,x_n^{(h)}\}) = g$ and the denomination of f is equal to m. Here, the *denomination* of f is the smallest number r such that $x_0^r p(x_1/x_0, \ldots, x_n/x_0) \in \mathcal{F}\{x_0, x_1, \ldots, x_n\}$[25]. Since all these characteristic numbers are definable for differential polynomials, $\mathbb{C}_{(n,n-1,h,g,m,n)}$ is a definable subset of $\mathbb{A}^{\binom{m+n(h+1)}{n(h+1)}}$. Hence, the differential Chow variety of index $(n-1, h, g, m)$ exists.

Remark 5.3. We remark here that if the Kolchin closure is proved to be definable in families, then the proof of the existence of differential Chow varieties could be greatly simplified and instead of using the algebraic Chow varieties to parameterize differential Chow varieties, one could directly show that the differential Chow coordinates of differential cycles of certain index constitute a differentially constructible set.

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Appendix: Geometric irreducibility and Zariski closure are definable in families by William Johnson

In this appendix we establish the results on definability in algebraically closed fields stated as Fact 2.22 in the main text. We assume that the readers are familiar with model theoretic notion and we follow standard model theoretic notations and conventions. For example, we write $RM(a/B)$ for the Morley rank of the type of a over B and use the nonforking symbol freely.

A.1. **Irreducibility in Projective Space.** Let $\mathbb{U}$ be a monster model of ACF. For $x \in \mathbb{P}^n(\mathbb{U})$, let $\mathbb{P}_x$ be the $(n-1)$-dimensional projective space of lines through x, and let $\pi_x : \mathbb{P}^n \setminus \{x\} \to \mathbb{P}_x$ be the projection.

Lemma A.1. *Let A be a small set of parameters, and suppose $x \in \mathbb{P}^n(\mathbb{U})$ is generic over A. Suppose V is an A-definable Zariski closed subset of $\mathbb{P}^n$, of codimension* greater than *1. Then $\pi_x(V) \subseteq \mathbb{P}_x$ is well-defined, Zariski closed, of codimension* *one less than the codimension of V. Moreover, $\pi_x(V)$ is irreducible if and only if V is irreducible.*

Proof. Replacing A with $\mathrm{acl}(A)$, we may assume A is algebraically closed, implying that the irreducible components of V are also A-definable.

Since x is generic, and V has codimension *at least* 1, $x \notin V$ so $\pi_x(V)$ is well-defined. It is Zariski closed because $\mathbb{P}^n$ is a complete variety, so V is complete and the image of V under any morphism of varieties is closed.

Claim A.2. *Let C be any irreducible component of V, and let $c \in V$ realize the generic type of C, over Ax. Then c is the sole preimage in V of $\pi_x(c)$.*

Proof. The generic type of C is A-definable, so $c \mathop{\smile\!\!\!\!\!\!|}_A x$, and therefore $RM(x/Ac) = RM(x/A) = n$. Suppose for the sake of contradiction that there was a second point $d \in V$, $d \neq c$, satisfying

$$\pi_x(d) = \pi_x(c).$$

This means exactly that the three points c, d, and x are colinear. Then x is on the 1-dimensional line determined by c and d, so

$$RM(x/Acd) \leq 1.$$

But then

$$n = RM(x/Ac) \leq RM(xd/Ac) = RM(x/Acd) + RM(d/Ac) \leq 1 + RM(V) < n,$$

by the codimension assumption. □

Using the claim, we see that $\pi_x(V)$ and V have the same dimension (= Morley rank). Indeed, let $v \in V$ have Morley rank $RM(V)$ over Ax. Then v realizes the generic type of *some* irreducible component C, so by the claim, v is interdefinable over Ax with $\pi_x(v)$. But then

$$RM(\pi_x(V)) \geq RM(\pi_x(v)/Ax) = RM(v/Ax) = RM(V),$$

and the reverse inequality is obvious. So the codimension of $\pi_x(V)$ is indeed one less.

Let $C_1, \ldots, C_m$ enumerate the irreducible components of V (possibly $m = 1$). Each of the components C_i is a closed subset of $\mathbb{P}^n$, and so by completeness each of the images $\pi_x(C_i)$ is a Zariski closed subset of $\mathbb{P}_x$. The image of each of the components is irreducible, on general grounds. If $\pi_x(C_i) \subseteq \pi_x(C_j)$ for some $i \neq j$, then the generic type of C_i would have the same image under π_x as some point in C_j, contradicting the Claim. So $\pi_x(C_i) \not\subseteq \pi_x(C_j)$ for $i \neq j$. It follows that the images $\pi_x(C_i)$ are the irreducible components of

$$\pi_x(V) = \bigcup_{i=1}^{m} \pi_x(C_i).$$

Therefore, $\pi_x(V)$ and V have the same number of irreducible components, proving the last point of the lemma. □

Theorem A.3. *Let $X_a \subseteq \mathbb{P}^n$ be a definable family of Zariski closed subsets of $\mathbb{P}^n$. Then the set of a for which X_a is irreducible, is definable.*

Proof. Dimension is definable in families, because ACF is strongly minimal. So we may assume that all (non-empty) X_a have the same (co)dimension. We proceed by induction on codimension, allowing n to vary.

For the base case of codimension one, we note the following:

(1) The family of Zariski closed subsets of $\mathbb{P}^n$ is ind-definable, that is a small (i.e. less than the size of the monster model) union of definable families, because the Zariski closed subsets are exactly the zero sets of finitely-generated ideals.
(2) Using 1, the family of *reducible* Zariski closed subsets of $\mathbb{P}^n$ is also ind-definable, because a definable set is a *reducible* Zariski closed set if and only if it is the union of two incomparable (with respect to containment) Zariski closed sets.
(3) Whether or not a polynomial in $\mathbb{C}[y_1, \ldots, y_{n+1}]$ is irreducible, is definable in terms of the coefficients, because we only need to quantify over lower-degree polynomials.
(4) A hypersurface in $\mathbb{P}^n$ is irreducible if and only if it is equal to the zero-set of an irreducible homogeneous polynomial. It follows by 3 that the family of irreducible codimension 1 closed subsets of $\mathbb{P}^n$ is ind-definable.
(5) By 2 (resp. 4), the set of a such that X_a is reducible (resp. irreducible) is ind-definable. Since these two sets are complementary, both are definable, proving the base case.

For the inductive step, suppose that irreducibility is definable in families of codimension one less than X_a. By choosing an isomorphism between $\mathbb{P}_x$ and $\mathbb{P}^{n-1}$, one easily verifies the definability of the set of (x, a) such that $\pi_x(X_a)$ is irreducible and has codimension one less.

By Lemma A.1, X_a is irreducible if and only if (x, a) lies in this set, for generic x. Definability of types in stable theories then implies definability of the set of a such that X_a is irreducible. $\square$

Corollary A.4. *The family of irreducible closed subsets of $\mathbb{P}^n$ is ind-definable.*

Proof. The family of closed subsets is ind-definable, and by Theorem A.3 we can select the irreducible ones within any definable family. $\square$

Corollary A.5. *The family of pairs $(X, \overline{X})$ with X definable and $\overline{X}$ its Zariski-closure, is ind-definable.*

Proof. By quantifier elimination in ACF, any definable set X can be written as a union of sets of the form $C \cap U$ with C closed and U open. Replacing V with a union of irreducible components, and distributing, we can write X as a union $\bigcup_{i=1}^m C_i \cap U_i$, with C_i Zariski closed and U_i Zariski open. We may assume that $C_i \cap U_i \neq \emptyset$ for each i, or equivalently, that $C_i \setminus U_i \neq C_i$.

In any topological space, closure commutes with unions, so

$$\overline{X} = \bigcup_{i=1}^n \overline{C_i \cap U_i}.$$

Now $\overline{C_i \cap U_i} \subseteq \overline{C_i} = C_i$, and

$$C_i = \overline{C_i \cap U_i} \cup (C_i \setminus U_i),$$

so by irreducibility of C_i, $\overline{C_i \cap U_i} = C_i$. Therefore,

$$\overline{X} = \bigcup_{i=1}^n C_i.$$

Corollary A.4 implies the ind-definability of the family of pairs

$$\left(\bigcup_{i=1}^{n} \overline{C_i \cap U_i}, \bigcup_{i=1}^{n} C_i\right)$$

with C_i irreducible closed, U_i open, and $C_i \cap U_i \neq \emptyset$. We have seen that this is the desired family of pairs. □

The following corollary is an easy consequence:

Corollary A.6. *Let X_a be a definable family of subsets of $\mathbb{P}^n$. Then the Zariski closures $\overline{X_a}$ are also a definable family.*

A.2. Irreducibility in Affine Space.

Theorem A.7. *Let X_a be a definable family of subsets of affine n-space.*

1. *The family of Zariski closures $\overline{X_a}$ is also definable.*
2. *The set of a such that $\overline{X_a}$ is irreducible is definable. More generally, the number of irreducible components of $\overline{X_a}$ is definable in families (and bounded in families).*
3. *Dimension and Morley degree of X_a are definable in a.*
4. *If each $\overline{X_a}$ is a hypersurface given by the irreducible polynomial $F_a(y_1, \ldots, y_n)$, then the degree of F_a in each y_i is definable in a. In fact, the polynomials F_a have bounded total degree and the family of F_a (up to scalar multiples) is definable.*
5. *The family of irreducible components of the Zariski closure is definable in families.*

Proof. (1) Embed $\mathbb{A}^n$ into $\mathbb{P}^n$. Then the Zariski closure of X_a within $\mathbb{A}^n$ is the intersection of $\mathbb{A}^n$ with the closure within $\mathbb{P}^n$. Use Corollary A.6.

(2) The number of irreducible components of the Zariski closure is the same whether we take the closure in $\mathbb{A}^n$ or $\mathbb{P}^n$. This proves the first sentence. The first sentence yields the ind-definability of the family of irreducible Zariski closed subsets of $\mathbb{A}^n$, from which the second statement is an exercise in compactness.

(3) We may assume X_a is closed, since taking the closure changes neither Morley rank nor Morley degree. The family of d-dimensional Zariski irreducible closed subsets of $\mathbb{A}^n$ is ind-definable, making this an exercise in compactness.

(4) Whether or not an n-variable polynomial is irreducible is definable in the coefficients, because to check reducibility one only needs to quantify over the (definable) set of lower-degree polynomials. This makes the family of irreducible polynomials ind-definable. Therefore, the set of pairs (a, F_a) where F_a cuts out $\overline{X_a}$, is ind-definable. For any given a, all the possibilities for F_a are essentially the same, differing only by scalar multiples. So the total degree of F_a only depends on a, and compactness yields a bound on the total degree. This in turn makes the set of pairs (a, F_a) definable.

(5) By [16, Proposition 3], every irreducible subvariety of $\mathbb{A}^n$ which is of codimension d is given (set-theoretically) by the intersection the zero sets of

$n+1$ polynomials, whose degrees are bounded by the degree of the variety.[2] Since the degree of a family of varieties is uniformly bounded by the product, D, of the degrees of the defining polynomials and the number of components is bounded by the degree, there are at most D many maximal irreducible subvarieties, each of which has degree less than or equal to D. Among such zero sets, the components of the variety are those which are maximal and irreducible.

□

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[2]By results of Storch [38] or Eisenbud and Evans [5] and a short argument, one can improve this to $d+1$ polynomials. Degree bounds are not given in [38] or [5]. For our purposes the existence of such bounds is what matters, so we will not pursue the details further, but merely note that the bounds of [16] are not tight in this case.